Archimedean Property

• (1) $\forall x \in \mathbb{Q}$, $\exists n_1 \in \mathbb{Z}$ s.t. $n_1 > x$

(1') $\forall x \in \mathbb{Q}$, $\exists n_2 \in \mathbb{Z}$ s.t. $n_2 < x$.

Pf of (1) Let $x = p/q$ $\quad p \in \mathbb{Z},\ q \in \mathbb{N}$.

then $\exists N_1$ s.t. $p < N_1$.

now one of $N_1, N_1+1, N_1+2, \ldots, N_1+q-1$
can be written as qn_1 $\exists n_1 \in \mathbb{Z}$

so $p < qn_1$ $\quad \exists n_1 \in \mathbb{Z}$

$x = \frac{p}{q} < n_1$ $\quad \exists n_1 \in \mathbb{Z}$.

(1') Apply (1) to $-x$.

• (2) $\forall x \in \mathbb{Q}$, $y \in \mathbb{Q}$ such that $x < y$.
$\exists p \in \mathbb{Q}$ s.t. $x < q < y$.

Pf of (2) By (1) $\exists N^{\in \mathbb{N}}$ s.t. $N > \frac{1}{y-x}$.

$\iff$ $y - x > \frac{1}{N}$ $\quad \exists N \in \mathbb{N}$.

$\iff$ $y > \underbrace{x + \frac{1}{N}}_{q} > x$ $\quad$ for some $x + \frac{1}{N} \in \mathbb{Q}$.

• (3) $\forall x \in \mathbb{Q}$, $\exists n_0 \in \mathbb{Z}$ s.t. $n_0 \le x < n_0 + 1$.

(3′) $\forall x \in \mathbb{Q}$, $y \in \mathbb{Q}$, $y > 0$, $\exists n_0$ s.t. $n_0 y \le x < (n_0+1) y$.

Pf of (3) Let $S = \{n \in \mathbb{Z} : n_2 \le n \le n_1 \text{ and } n \le x\}$.

where n_1, n_2 are given by (1) (1′).

Then S is nonempty (since $n_2 \in S$), and S is finite.

These allow us to take $n_0 = \max S$.

Now $n_0 < n_1$ as $n_1 \notin S$ (because $n_1 > x$).

so $n_0 + 1 \le n_1$, and

$n_0 + 1 > x$ (otherwise $n_0+1 \in S$ and $n_0 \ne \max S$)

$\Longrightarrow$ $n_0 \le x$ (as $n_0 \in S$) and $n_0 + 1 > x$

#.

Proof of (3′): Apply (3) to $\frac{x}{y}$ #.

$$\boxed{\text{Proof of } "\tfrac{1}{\alpha} \in R" \text{ and } "\alpha\left(\tfrac{1}{\alpha}\right) = 1^*"}$$

Let $\alpha \in R^+ =, \{\beta \in R : p \in \beta$ for some positive $p\}$.

Define $\frac{1}{\alpha} = \{p \in Q : p+s < \frac{1}{r}$ for some $s>0$, $r \notin \alpha$, $r>0\}$.

<u>Claim</u> $\frac{1}{\alpha} \in R^+$

(I) (nonempty) (Contains a positive rational #) Let $r_1 \notin \alpha$ (α is proper subset of Q).

then $r_1 > 0$ (as $\alpha \in R^+$).

and $p = \frac{1}{3r_1}$ satisfies.

$p > 0$, $\quad p + \underbrace{\frac{1}{3r_1}}_{>0} < \frac{1}{r_1}$ $\quad \exists r_1 \notin \alpha$.

Hence $p \in \left(\frac{1}{\alpha}\right)$.

($\frac{1}{\alpha} \subsetneq Q$) Let $r_2 > 0$, $r_2 \in \alpha$. We claim that $\frac{1}{r_2} \notin \frac{1}{\alpha}$.

Otherwise $\frac{1}{r_2} \in \frac{1}{\alpha}$.

and $\frac{1}{r_2} < \frac{1}{r_2} + s < \frac{1}{r}$ $\quad \exists r \notin \alpha$.

but then $r < r_2$ $\quad \exists r \notin \alpha$, $r_2 \in \alpha$.

which is a contradiction.

<u>Direct proof.</u> $\forall r > 0$, $r \notin \alpha$

$r_2 < r$ (since $r_2 > 0$, $r_2 \in \alpha$)

$\frac{1}{r_2} > \frac{1}{r}$ (since $r, r_2 > 0$).

$\frac{1}{r_2} + s > \frac{1}{r}$ $\quad \forall s > 0$, $\forall r \notin \alpha$.

$\Rightarrow \frac{1}{r_2} \notin \frac{1}{\alpha}$.

So $\frac{1}{\alpha} \subsetneq Q$.

(II) Let $p \in \frac{1}{\alpha}$, and $q < p$

then $q + s < p + s < \frac{1}{r}$ $\quad \exists\, s > 0,\ r > 0,\ r \notin \alpha$.

Here $q \in \frac{1}{\alpha}$.

(III) $p \in \frac{1}{\alpha}$ then $p + s < \frac{1}{r}$ $\quad \exists\, s > 0,\ r \notin \alpha$.

then $\left(p + \frac{s}{2}\right) + \frac{s}{2} < \frac{1}{r}$ $\quad \exists\, \frac{s}{2} > 0,\ r \notin \alpha$.

Here $\left(p + \frac{s}{2}\right) \in \frac{1}{\alpha}$. with $p + \frac{s}{2} > p$

Claim $\alpha\left(\frac{1}{\alpha}\right) \subset 1^* = \{p \in \mathbb{Q} : p < 1\}$.

Let $p \in \alpha\left(\frac{1}{\alpha}\right)$, then $p < rq$ $\quad \exists\, r \in \alpha,\ q \in \frac{1}{\alpha},\ r, s > 0$.

$q \in \frac{1}{\alpha} \Rightarrow q < \frac{1}{r'} - s$ $\quad \exists\, s, r'$ positive, $r' \notin \alpha$.

$\Rightarrow p < r\left(\frac{1}{r'} - s\right) < \frac{r}{r'} < 1$

(since r, r' positive, $r \in \alpha$, $r' \notin \alpha \Rightarrow 0 < r < r'$).

$\Rightarrow \alpha\left(\frac{1}{\alpha}\right) < 1^*$

<u>Claim</u> $1^* \subset \alpha\left(\frac{1}{\alpha}\right)$.

Let $p \in 1^*$ i.e. $p < 1$.

By Archimedean Property, $\exists\, n_0 \in \mathbb{N}$ s.t. $n > \frac{3}{1-p} \Leftrightarrow p + \frac{3}{n} < 1$ $\forall n \geq n_0$.

Let $r_0 \in \alpha$ and $r_0 > 0$

<u>Claim</u> $\exists\, \bar{n} \geq n_0$ s.t. $\frac{\bar{n} r_0}{n_0} \in \alpha$, $\frac{(\bar{n}+1) r_0}{n_0} \notin \alpha$.

Existence of $\bar{n} \in \mathbb{Z}$ follows from Archimedean property.

That $\bar{n} \geq n_0$ follows from

$$\underbrace{\frac{(\bar{n}+1) r_0}{n_0}}_{\notin \alpha} > \underbrace{r_0}_{\in \alpha} \iff \bar{n}+1 > n_0 \iff \bar{n} \geq n_0$$

Hence $p < \frac{(\bar{n}+1) - 3}{\bar{n}+1} = \frac{\bar{n} r_0}{n_0} \left(\frac{1}{\frac{(\bar{n}+1) r_0}{n_0}} - \frac{2 n_0}{\bar{n}(\bar{n}+1) r_0} \right)$

where $\frac{\bar{n} r_0}{n_0} \in \alpha$, $\frac{(\bar{n}+1) r_0}{n_0} \notin \alpha$

and $\left(\frac{1}{\frac{(\bar{n}+1) r_0}{n_0}} - \frac{2 n_0}{\bar{n}(\bar{n}+1) r_0} \right) < \underbrace{\frac{1}{\left(\frac{(\bar{n}+1) r_0}{n_0}\right)}}_{\frac{1}{r'},\ r' \notin \alpha} - \underbrace{\frac{n_0}{\bar{n}(\bar{n}+1) r_0}}_{s > 0}$.

$$\Rightarrow \left(\frac{1}{\frac{(\bar{n}+1) r_0}{n_0}} - \frac{2 n_0}{\bar{n}(\bar{n}+1) r_0} \right) \in \frac{1}{\alpha}.$$

$$\Rightarrow p < \underbrace{\frac{\bar{n} r_0}{n_0}}_{\in \alpha} \underbrace{\left(\cdots \right)}_{\notin \alpha} \longrightarrow p \in \alpha\left(\frac{1}{\alpha}\right).$$

$$\Rightarrow 1^* \subset \alpha\left(\frac{1}{\alpha}\right).$$

Distributive Law in R^+

To show $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

Let $p \in \alpha(\beta+\gamma)$, then $p < r(s_1+s_2)$ $\quad r\in\alpha$, $s_1\in\beta$, $s_2\in\gamma$.

$\Rightarrow p < rs_1 + rs_2$ $\quad$ all positive.

Let $r' > r$, $r' \in \alpha$.

then $rs_1 < r's_1$ $\quad r'\in\alpha$, $s_1\in\beta$

$\Rightarrow rs_1 \in \alpha\beta$.

Similarly $rs_2 \in \alpha\gamma$.

Hence $p < rs_1 + rs_2$, the latter lies in $\alpha\beta+\alpha\gamma$.

Hence $p \in \alpha\beta+\alpha\gamma$ $\quad \forall p \in \alpha(\beta+\gamma)$.

$\Longrightarrow \alpha(\beta+\gamma) \subset \alpha\beta+\alpha\gamma$.

Alternatively, let $p \in \alpha\beta+\alpha\gamma$

then $p = p_1 + p_2$, $\quad p_1 < r_1 s_1$ $\quad r_1\in\alpha$, $s_1\in\beta$, positive.

$p_2 < r_2 s_2$ $\quad r_2\in\alpha$, $s_2\in\gamma$ positive.

So $p = p_1+p_2 < r_1s_1 + r_2s_2 < \underbrace{\max\{r_1,r_2\}}_{\in\alpha}\cdot\underbrace{(s_1+s_2)}_{\in\beta+\gamma}$.

$\Longrightarrow p \in \alpha(\beta+\gamma)$ $\quad \forall p \in \alpha\beta+\alpha\gamma$

$\Rightarrow \alpha\beta+\alpha\gamma \subset \alpha(\beta+\gamma)$
