1.28 Theorem. *The interval* $(0,1)$ *in* $\mathbf{R}$ *is not countable.*

Proof. Let E be a countable subset of $(0,1)$, say $E = \{x_1, x_2, \ldots\}$. We make the following practically trivial remark: given a nonempty open interval (a,b) in $\mathbf{R}$, and $x \in \mathbf{R}$, there exist $c < d$ such that $[c,d] \subset (a,b)$ and $x \notin [c,d]$. By this remark, we can choose $a_1 < b_1$ such that $x_1 \notin [a_1, b_1] \subset (0,1)$. Having chosen $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ such that $a_k < b_k$ and $x_k \notin [a_k, b_k]$ for $k = 1, 2, \ldots, n$, and such that $[a_{k+1}, b_{k+1}] \subset (a_k, b_k)$ for $k = 1, \ldots, n-1$, the remark enables us to choose $a_{n+1} < b_{n+1}$ such that $[a_{n+1}, b_{n+1}] \subset (a_n, b_n)$ and $x_{n+1} \notin [a_{n+1}, b_{n+1}]$. Thus we have inductively defined for each $n \in \mathbf{N}$ a closed interval $J_n = [a_n, b_n]$, such that $J_{n+1} \subset J_n$ and $x_n \notin J_n$ for every $n \in \mathbf{N}$. According to Theorem 1.27, there exists $x \in \bigcap_{n=1}^{\infty} J_n$. Since $x_m \notin \bigcap_{n=1}^{\infty} J_n$ for every $m \in \mathbf{N}$, we conclude $x \notin E$. Thus $E \neq (0,1)$. ∎

1.8 Algebraic and Transcendental Numbers

1.29 Definition. *A real number* x *is said to be* algebraic *if there exists a positive integer* n, *and integers* $a_0, a_1, \ldots, a_n$, $a_n \neq 0$, *such that*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{1.1}$$

We say that x *is* algebraic *of* degree n *if* n *is the smallest positive integer for which* x *satisfies an equation of the form* (1.1). *We say that* x *is* transcendental *if it is not algebraic.*

We were motivated to expand from $\mathbf{Q}$ to $\mathbf{R}$ in order to be able to solve the equation $x^2 = 2$. We found that in $\mathbf{R}$ we could solve any equation $x^n = a$, but that still leaves open the possibility that every real number is algebraic. There are at least two ways to see that this is not so.

1.30 Proposition. *The set of transcendental numbers is uncountable.*

It is perhaps a little disappointing that this existence proof for transcendental numbers fails to exhibit a single one. Here is another approach. The following theorem of Liouville says that algebraic numbers which are not rational cannot be approximated too closely by rational numbers.

1.31 Theorem. *Let* x *be an algebraic number of degree not more than* n. *Then there exists a constant* $C > 0$ *such that for any integers* p, q $(q > 0)$ *with* $x \neq p/q$, *we have*

$$\left| x - \frac{p}{q} \right| > \frac{C}{q^n}.$$

Proof. Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$, where a_j is an integer for $j = 0, 1, \ldots, n$, such that $f(x) = 0$. Then $f(t) = (t - x) g(t)$, where g is a polynomial of degree less than n (with real coefficients). Since g has at most $n-1$ zeros, there exists $\delta > 0$ such that $0 < |t - x| \leq \delta$ implies $g(t) \neq 0$, and hence also $f(t) \neq 0$. It is easy to see that there exists M such that $|g(t)| \leq M$ for all t with $|t - x| \leq \delta$. For instance, if $g(t) = \sum_{j=0}^{n-1} b_j t^j$, choose N large enough so that $[x - \delta, x + \delta] \subset [-N, N]$, and take $M = \sum_{j=0}^{n-1} |b_j| N^j$. Now suppose $|x - p/q| \leq \delta$, $x \neq p/q$. Then $g(p/q) \neq 0$, so we have

$$\frac{p}{q} - x = \frac{f(p/q)}{g(p/q)} = \frac{a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n}{q^n g(p/q)}.$$

Now the numerator of this last fraction is an integer, and it is not 0, since $f(t) \neq 0$ for $0 < |t - x| \leq \delta$. Hence the numerator has absolute value at least 1, and we have

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{M q^n}.$$

But if $|x - p/q| > \delta$, then of course $|x - p/q| > \delta / q^n$, since $q \geq 1$. Thus taking any C smaller than both δ and $1/M$ gives us the theorem. ∎

1.32 Example. Let $x_1 = \frac{1}{2}$, and inductively define

$$x_n = x_{n-1} + \frac{1}{2^{n!}}.$$

Then $x_{n+m} - x_n = 2^{-(n+1)!} + \cdots + 2^{-(n+m)!} < 2 \cdot 2^{-(n+1)!}$ for every n, m. Let $x = \sup\{x_1, x_2, \ldots\}$. We have then $x - x_n \leq 2 \cdot 2^{-(n+1)!}$. But $x_n = p_n 2^{-n!}$ for some integer p. Let $q_n = 2^{n!}$, so $2^{(n+1)!} = q_n^{n+1}$. We have then the inequalities $0 < x - p_n/q_n = x - x_n \leq 2 \cdot 2^{-(n+1)!} = (2/q_n)/q_n^n$ for every n. According to Theorem 1.31, this is impossible if x is algebraic, so x is transcendental.

1.9 Existence of R

In this section, we outline the construction first given by Dedekind. Dedekind presented his real numbers as "cuts" of the line, i.e., as pairs of sets of rationals, one set lying entirely to the left of the other, the union being the set of all rationals. Nowadays, we dispense with one element of the