Math 2255, Spring 2017, Midterm 1, Feb 15 **Name:** ______________________

1. [10] Solve the differential equation $y' + y = e^{-t}$, $y(0) = 0$.

<u>Answer:</u> The integrating factor satisfies $\frac{\mu'}{\mu} = 1$. By inspection, we choose $\mu(t) = e^t$.

$$(e^t y)' = e^t(y' + y) = e^t e^{-t} = 1.$$

Integrate from 0 to t,

$$e^t y(t) - e^0 y(0) = t$$

and hence using $y(0) = 0$, we have

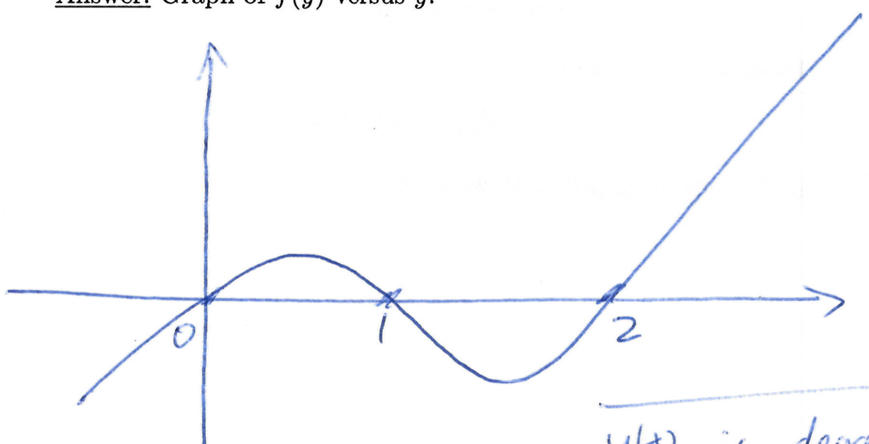
$$y(t) = te^{-t}.$$

2. [15] Consider the autonomous equation

$$\frac{dy}{dt} = y(y-1)(y-2), y(0) = y_0$$

where y_0 is a non-negative constant. Sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable.

Answer: Graph of $f(y)$ versus y:



$y(t)$ is decreasing when $y<0$ or $1<y<2$

$y(t)$ is increasing when $0<y<1$ and $y>2$.

So the equilibria (and their stability) are :

- y=0; unstable
- y=1; stable
- y=2; unstable

3. [20] Consider the differential equation $y + (2x - y)y' = 0$.

   (a) (10 points) Show that the following equation is not exact, but becomes exact when multiplied by an integrating factor in the form of $\mu(y)$, a function of y only.

   (b) (10 points) Find the equation for its integral curves. You may leave the answer in implicit form.

Answer: $M(x, y) = y$, $N(x, y) = (2x - y)$.

$$M_y - N_x = 1 - 2 \neq 0 \quad \Longrightarrow \quad \text{not exact.}$$

Multiply the DE by the integrating factor $\mu(y)$, a function of y only: $y\mu(y) + \mu(y)(2x - y)y' = 0$. So

$$\tilde{M}(x, y) = y\mu(y) \quad N(x, y) = \mu(y)(2x - y)$$

and $\tilde{M}_y - \tilde{N}_x = y\mu'(y) + \mu(y) - 2\mu(y) = y\mu'(y) - \mu(y)$. Hence we need to choose $\mu(y)$ so that

$$\frac{\mu'}{\mu} = \frac{1}{y} \quad \Longleftrightarrow \quad \frac{d}{dy}(\ln \mu) = \frac{1}{y}.$$

By inspection, we can choose $\mu(y) = y$. Hence the DE is exact after multiplying by y

$$y^2 + (2xy - y^2)y' = 0. \tag{1}$$

Sine the DE (1) is exact, he integral curves are given by $\varphi(x, y) = C$, where $\varphi_x = y^2$ and $\varphi_y = 2xy - y^2$. From the latter we have

$$\varphi(x, y) = xy^2 - \frac{y^3}{3} + h(x) \quad \Longrightarrow \quad \varphi_x = y^2 + h'(x).$$

Since $\varphi_x = y^2$, we see that $h'(x) = 0$ and we may choose $h(x) = 0$. Hence $\varphi(x, y) = xy^2 + \frac{y^3}{3}$, and the integral curves are given by

$$xy^2 - \frac{y^3}{3} = C.$$

4. [15] Solve the given initial value problem, and describe its behavior for increasing t.

$$y'' - 6y' + 9y = 0, \quad y(0) = 0, \; y'(0) = 2.$$

Answer: Solving the characteristic equation

$$r^2 - 6r + 9 = (r-3)^2 = 0 \quad \Longrightarrow \quad r_1 = r_2 = 3.$$

We have $y_1(t) = e^{3t}$. In general, let $y(t) = v(t)e^{3t}$, then

$$\begin{aligned} y'' - 6y' + 9y &= (v''e^{3t} + 6v'e^{3t} + 9ve^{3t}) - 6(v'e^{3t} + 3ve^{3t}) + 9ve^{3t} \\ &= e^{3t}[v'' + 6v' + 9v - 6v' - 18v + 9v] = e^{3t}v'' \end{aligned}$$

Hence $v'' = 0$ and we deduce that $v(t) = c_1 t + c_2$ and $y(t) = e^{3t}(c_1 t + c_2)$. It remains to solve for c_1 and c_2.

$$y(0) = 0 \quad \Longrightarrow \quad c_2 = 0 \quad \text{and} \quad y(t) = c_1 t e^{3t}.$$

Hence $y'(t) = c_1(1 - 3t)e^{3t}$ and

$$y'(0) = 2 \quad \Longrightarrow \quad c_1 = 2.$$

Hence $y(t) = 2te^{3t}$. $y(t) \to \infty$ as t increases.

5. [10] Consider the first order difference equation $y_{n+1} = f(y_n)$, where $f(s) = 1 - 2s$. Find $y_1, y_2, ..., y_5$ in terms of y_0 and also y_n in terms of y_0.

<u>Answer:</u>

$$
\begin{aligned}
y_1 &= 1 - 2y_0 \\
y_2 &= 1 - 2(1 - 2y_0) = 1 - 2 + 2^2 y_0 \\
y_3 &= 1 - 2(1 - 2 + 2^2 y_0) = 1 - 2 + 2^2 - 2^3 y_0 \\
y_4 &= 1 - 2(1 - 2 + 2^2 - 2^3 y_0) = 1 - 2 + 2^2 - 2^3 + 2^4 y_0 \\
y_5 &= 1 - 2(1 - 2 + 2^2 - 2^3 + 2^4 y_0) = 1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 y_0
\end{aligned}
$$

Hence, the general formula for y_n reads

$$
\begin{aligned}
y_n &= 1 - 2 + 2^2 - 2^3 + \cdots + (-2)^{n-1} + (-2)^n y_0 \\
&= \frac{(-2)^n - 1}{-2 - 1} + (-2)^n y_0 \\
&= \frac{1 - (-2)^n}{3} + (-2)^n y_0
\end{aligned}
$$