Name: Instructor

MATH 2255 ODEs and their Applications
SPRING 2017 MIDTERM 2, March 24, 2017

Each problem counts for 20 points.

No books, notes, or calculators are allowed.

Problem	Score
1	
2	
3	
4	
Total	

(1) [15pt] Find the solutions of the given initial value problem

$$y'' + y = g(t), \quad y(0) = 0,\ y'(0) = 1; \quad g(t) = \begin{cases} t/2, & \text{for } 0 \le t < 6, \\ 3 & \text{for } 6 \le t < \infty. \end{cases}$$

and sketch the graph of the forcing function.

$$y'' + y = \frac{t}{2} + u_6(t)\left[-\frac{t}{2} + 3\right] = \frac{t}{2} - \frac{1}{2}u_6(t)(t-6)$$

Take the Laplace transform:

$$(s^2+1)Y(s) - 1 = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2}H(s) - \frac{e^{-6s}}{2}H(s)$$

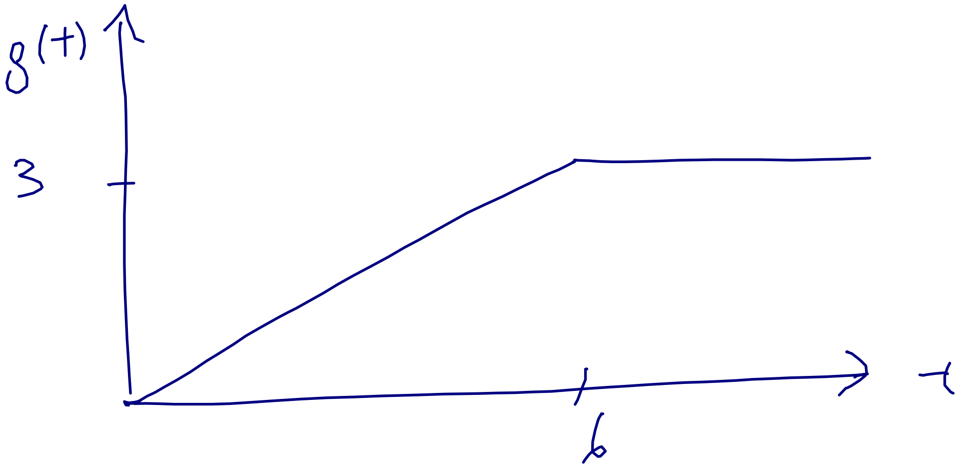
where $H(s) = \dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

and $h(t) = \mathcal{L}^{-1}\{H(s)\} = t - \sin t$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = \sin t + \frac{1}{2}h(t) - \frac{1}{2}u_6(t)h(t-6)$$

$$= \sin t + \frac{1}{2}(t - \sin t) - \frac{1}{2}u_6(t)\left[(t-6) - \sin(t-6)\right]$$

Forcing fcn. $g(t)$



(2) (a)[6pt] Solve the initial value problem

$$y'' + 4y = 3\cos t, \quad y(0) = 0, \quad y'(0) = 0,$$

[The answer should be in the form of a difference of two $cos(\omega t) - \cos(\omega_0 t)$. If your answer doesn't match, try again.]

(b) [9pt] Write the solution $y(t)$ as a product of two trigonometric functions of different frequencies. [Hint: Make use of the trigonometric identities

$$\cos(A+B) = \cos A \cos B - \sin A \sin B, \quad \cos(A-B) = \cos A \cos B + \sin A \sin B,$$

with $A = (\omega_0 + \omega)t/2$ and $B = (\omega_0 - \omega)t/2$.]

(a) By Method of undetermined coeff.:

$y(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 \cos t + C_4 \sin t$.

$y'' + 4y = 3C_3 \cos t + 3C_4 \sin t \Rightarrow C_3 = 1, \ C_4 = 0$.

$\Rightarrow y(t) = C_1 \cos 2t + C_2 \sin 2t + \cos t$

$y'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t - \sin t$

$\Rightarrow \quad 0 = y(0) = C_1 + 1$

$\quad\ \ 0 = y'(0) = 2C_2$

$\Rightarrow C_1 = -1$

$\quad\ \ C_2 = 0$.

$\Rightarrow y(t) = \cos t - \cos 2t$.

(b) Take $A = \frac{3t}{2}$ and $B = \frac{t}{2}$

$y(t) = \cos(A-B) - \cos(A+B)$

$= 2 \sin A \sin B$

$= 2 \sin \frac{3t}{2} \sin \frac{t}{2}$

(3) (a) [10pt] Solve the integro-differential equation.

$$\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\, d\xi = t, \quad \phi(0) = 0.$$

(b) [5pt] By differentiating the above integro-differential equation a sufficient number of times, convert it into an initial value ODE problem.

(a) Take Laplace transform: Let $\Phi(s) = \mathcal{L}\{\phi(t)\}$:

$$\mathcal{L}\{\phi'(t)\} + \mathcal{L}\{t * \phi(t)\} = \mathcal{L}\{t\}.$$

$$s\mathcal{L}\{\phi(t)\} + \mathcal{L}\{t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{t\}.$$

$$s\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{1}{s^2}$$

$$\Phi(s) = \frac{1/s^2}{s + 1/s^2} = \frac{1}{s^3+1}$$

By method of partial fraction:

$$\Phi(s) = \frac{1}{(s+1)(s^2-s+1)} = \frac{a}{s+1} + \frac{bs+c}{s^2-s+1} = \frac{(a+b)s^2+(b+c-a)s+(a+c)}{(s+1)(s^2-s+1)}$$

$$\Rightarrow \begin{cases} a+b=0 \\ b+c-a=0 \\ a+c=1 \end{cases} \Rightarrow a = \frac{1}{3},\ b = -\frac{1}{3},\ c = \frac{2}{3}$$

$$\Rightarrow \Phi(s) = \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{s-2}{s^2-s+1}$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left[\frac{(s-\frac{1}{2}) - 3/2}{(s-\frac{1}{2})^2 + 3/4}\right].$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left[\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+3/4} - \sqrt{3}\,\frac{\sqrt{3}/2}{(s-\frac{1}{2})^2+3/4}\right]$$

$$\Rightarrow \phi(t) = \frac{1}{3}e^{-t} - \frac{1}{3}\left[e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

(b) Differentiating: (with respect to t)

$$\phi''(t) + \frac{d}{dt}\left[\int_0^t (t-\xi)\phi(\xi)\,d\xi\right] = 1.$$

$$\phi''(t) + (t-t)\phi(t) + \int_0^t \frac{d}{dt}\left[(t-\xi)\phi(\xi)\right]d\xi = 1$$

$$\phi''(t) + \int_0^t \phi(\xi)\,d\xi = 1$$

Differentiating with respect to t again:

$$\phi'''(t) + \phi(t) = 0.$$

Initial value: $\begin{cases} \phi(0)=0 \quad \text{(is given)}. \\ \phi'(t) = t - \int_0^t (t-\xi)\phi(\xi)\,d\xi \\ \quad \Rightarrow \phi'(0)=0. \\ \phi''(t) = 1 - \int_0^t \phi(\xi)\,d\xi \\ \quad \Rightarrow \phi''(0) = 1. \end{cases}$

$$\Rightarrow \boxed{\begin{cases} \phi'''(t) + \phi(t) = 0 \\ \phi(0) = \phi'(0) = 0, \quad \phi''(0) = 1 \end{cases}}$$

(4) Consider the initial value problem

$$y'' + \gamma y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0,$$

where $\gamma = 1/2$ is the deamping coefficient (or resistence).

(a) [9pt] Find the solution of the initial value problem and sketch its graph.

(b) [6pt] Find the time t_1 at which the solution attains its maximum value.

(a) Set $\gamma = 1/2$ and take the Laplace transform:

$$\left(s^2 + \tfrac{1}{2}s + 1\right)Y(s) = e^{-s}$$

$$Y(s) = e^{-s} H(s)$$

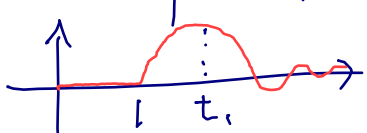
where $H(s) = \dfrac{1}{s^2 + \frac{1}{2}s + 1} = \dfrac{1}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}} = \dfrac{4}{\sqrt{15}} \dfrac{\sqrt{15}/4}{\left(s+\frac{1}{4}\right)^2 + 15/16}$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{4}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_1(t)\, h(t-1)$$

$$= u_1(t) \frac{4}{\sqrt{15}} e^{-\frac{(t-1)}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right)$$

(b) The maximum pt. t_1 is the first critical pt. in $\{t > 1\}$

as . For $t > 1$:

$$y'(t) = \frac{4}{\sqrt{15}} e^{-\frac{(t-1)}{4}} \left(-\frac{1}{4} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) + \frac{\sqrt{15}}{4} \cos\left(\frac{\sqrt{15}}{4}(t-1)\right) \right)$$

$$y'(t_1) = 0 \Rightarrow -\frac{1}{4} \sin\left(\frac{\sqrt{15}}{4}(t_1-1)\right) + \frac{\sqrt{15}}{4} \cos\left(\frac{\sqrt{15}}{4}(t_1-1)\right) = 0$$

$$\Rightarrow \tan\left(\frac{\sqrt{15}}{4}(t_1-1)\right) = \sqrt{15}$$

$$\Rightarrow t_1 = 1 + \frac{4}{\sqrt{15}} \tan^{-1}\left(\sqrt{15}\right)$$