(1) (a) [8pt] Determine intervals in which solutions for the following differential equation are sure to exist:

$$t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$$

$$y^{(4)} + \frac{e^t}{t(t-1)} y'' + \frac{4t}{t-1} y = 0$$

Solutions are sure to exist in

$(-\infty, 0)$, $(0, 1)$, $(1, \infty)$.

(b) [12pt] Determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

$$f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$$

Suppose $a(2t-3) + b(t^3+1) + c(2t^2-t) + d(t^2+t+1) = 0$

$\Leftrightarrow \ bt^3 + (2c+d)t^2 + (2a-c+d)t + (-3a+b+d) = 0$

$\Leftrightarrow \begin{cases} b = 0, & 2a-c+d=0 \\ 2c+d=0, & -3a+b+d=0 \end{cases} \Leftrightarrow \begin{cases} b=0 & 2a-c+d=0 \\ 2c+d=0 & -3a+d=0 \end{cases}$

$\Leftrightarrow \begin{cases} b=0 & 2a+\frac{3}{2}d=0 \\ 2c+d=0 & -3a+d=0 \end{cases}$

$\Leftrightarrow \begin{cases} a=b=d=0 \\ 2c+d=0 \end{cases} \Rightarrow a=b=c=d=0$

Hence f_1, f_2, f_3, f_4 are linearly independent.

(2) [20pt] Find the solution of the given initial value problem. How does the solution behave as $t \to \infty$?

$$4y''' + y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 8, \quad y''(0) = -1.$$

Charateristic Eq; $4r^3 + r + 5 = 0$

$(r = -1)$ is a root $\Rightarrow (r+1)(4r^2 - 4r + 5) = 0$

$\Rightarrow (r+1)\left[r^2 - r + \frac{5}{4}\right] = (r+1)\left[(r - \frac{1}{2})^2 + 1\right] = 0$

$\Rightarrow r = -1, \ \frac{1}{2} \pm i.$

General sol $\quad y(t) = C_1 e^{-t} + e^{+\frac{1}{2}t}(C_2 \cos t + C_3 \sin t)$

$y'(t) = -C_1 e^{-t} + e^{\frac{1}{2}t}\left[\left(\frac{C_2}{2} + C_3\right)\cos t + \left(-C_2 + \frac{C_3}{2}\right)\sin t\right]$

$-1 = y''(0) = C_1 + \left[\frac{C_2}{4} + \frac{C_3}{2} - C_2 + \frac{C_3}{2}\right] = C_1 - \frac{3C_2}{4} + C_3$

$2 = y(0) = C_1 + C_2 \quad \Rightarrow \quad C_1 = -C_2 + 2$

$8 = y'(0) = -C_1 + \frac{C_2}{2} + C_3$

$\Rightarrow \begin{cases} -\frac{7C_2}{4} + C_3 = -3 \\ \frac{3C_2}{2} + C_3 = 10 \end{cases} \quad \Rightarrow \quad \frac{13C_2}{4} = 13 \Rightarrow C_2 = 4$

$C_3 = 4, \quad C_1 = -2$

$\Rightarrow \quad y(t) = -2e^{-t} + e^{t/2}\left[4\cos t + 4\sin t\right]$

(3) Find the general solution of the given differential equation.

$$y''' - y'' - y' + y = e^t - 3$$

Characteristic Eq $\quad r^3 - r^2 - r + 1 = 0$

$\Leftrightarrow (r^2-1)(r-1) = 0 \Leftrightarrow (r-1)^2(r+1) = 0$

$\Rightarrow y_c(t) = (C_1 + C_2 t)e^t + C_3 e^{-t}$

Particular sol $\quad Y(t) = At^2e^t + B.$

$Y'(t) = Ae^t(t^2+2t)$

$Y''(t) = Ae^t(t^2+4t+2)$

$Y'''(t) = Ae^t(t^2+6t+6).$

$L[Y] = Ae^t[(2t+4) - 2t] + B = e^t - 3$

$\Rightarrow A = 1/4, \quad B = -3.$

$\Rightarrow y(t) = y_c(t) + Y(t) = (C_1 + C_2 t)e^t + C_3 e^{-t} + \frac{1}{4}t^2e^t - 3$

(4) [20] Determine a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$. [Hint: First show that $a_n = 0$ when n is odd. Next, write the recursive relation for a_{2k} $(k = 0, 1, 2, 3, 4...)$]

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} 2 a_{n-1} x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} + 2 a_{n-1} \right] x^n = 0$$

$\Rightarrow \quad a_1 = 0, \quad$ and $\quad (n+1) a_{n+1} + 2 a_{n-1} = 0$ for $n \geq 1$.

$\Rightarrow \quad a_1 = a_3 = a_5 \cdots = 0, \quad$ and,

$$a_{2k} = \frac{-2 a_{2k-2}}{2k} = -\frac{a_{2k-2}}{k} = \frac{a_{2k-4}}{k(k-1)} = \cdots = \frac{(-1)^k a_0}{k!}$$

$$\left(\text{e.g. } a_2 = -a_0, \quad a_4 = \frac{-a_2}{2} = \frac{a_0}{2 \cdot 1}, \quad a_6 = \frac{-a_0}{3 \cdot 2 \cdot 1} \right).$$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} = e^{-x^2}$$

(5) [20] Seek power series solution of the given differential equation of the given point x_0. (a) Find the recurrence relation. (b) Find the first four terms in each of two solutions y_1 and y_2.

$$y'' + xy' + 2y = 0, \quad x_0 = 0.$$

1(a) Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$\Rightarrow 2a_2 + 2a_0 = 0$, $\cancel{(n+2)}(n+1)a_{n+2} + \cancel{(n+2)}a_n = 0$, $n \geq 1$

(b) y_1: $a_2 = -a_0$, $a_4 = -\frac{a_2}{3} = \frac{a_0}{3}$

$a_6 = \frac{-a_4}{5} = \frac{+a_2}{5 \cdot 3} = -\frac{a_0}{5 \cdot 3 \cdot 1}$, $a_8 = \frac{a_0}{7 \cdot 5 \cdot 3 \cdot 1}$.

$y_1(t) = a_0\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{5 \cdot 3 \cdot 1} + \cdots\right)$.

y_2: $a_3 = -\frac{a_1}{2}$, $a_5 = -\frac{a_3}{4} = \frac{a_1}{4 \cdot 2}$ $a_7 = -\frac{a_1}{6 \cdot 4 \cdot 2}$.

$\Rightarrow y_2(t) = a_1\left(x - \frac{x^3}{2} + \frac{x^5}{4 \cdot 2} - \frac{x^7}{6 \cdot 4 \cdot 2} + \cdots\right)$.