Math 2255, Spring 2017, Quiz 2, Feb 1 **Name:** ______________________________

1. [10] Determine the interval of definition of the solution to the following differential equation.

$$\frac{dy}{dx} = \frac{x^3 - 2y}{x}, \quad y(1) = 1.$$

Answer: Rewrite the DE as $\frac{dy}{dx} + \frac{2}{x}y = x^2$. The DE is linear. Hence, by Theorem 2.4.1, the solution exists in the interval where $p(x) = \frac{2}{x}$ and $g(x) = x^2$ are continuous. The maximal interval for which $p(x)$ and $g(x)$ are continuous, and which includes the point $x_0 = 1$, is

$$0 < x < \infty.$$

2. [10] Use Euler's method with $h = 0.1$ to compute the approximate values of the solution at $t = 0.1, 0.2$.

$$y' - y = 1, \qquad y(4) = -2 + \sqrt{3}.$$

Answer:

$$f(y) = 1 + y, \quad h = 0.1, \quad t_0 = 0, \quad t_1 = 0.1, \quad t_2 = 0.2.$$

And by Euler's method with stepsize $h = 0.1$:

$$\begin{aligned} y_0 &= y(0) = -2, \\ y(0.1) \approx y_1 &= y_0 + f(y_0)h = -2 + (1-2)(0.1) = -2.1, \\ y(0.2) \approx y_2 &= y_1 + f(y_1)h = -2.1 + (1-2.1)(0.1) = -2.21. \end{aligned}$$

3. [10] Solve the given differential equation, and solve for an explicit formula for y in terms of x.

$$\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}, \qquad y(0) = 1.$$

Answer: Since it is not linear and not (clearly) separable, we check whether it is an exact equation. TO this end, rewrite the DE as

$$\underbrace{(2xy + y^2 + 1)}_{M(x,y)} + \underbrace{(x^2 + 2xy)}_{N(x,y)} \frac{dy}{dx} = 0.$$

Now,

$$M_y = 2x + 2y \quad \text{and} \quad N_x = 2x + 2y$$

are equal, so the DE is exact. Next, we need to find $\varphi(x, y)$ so that

$$\varphi_x(x, y) = M(x, y) = 2xy+y^2+1, \quad \text{and} \quad \varphi_y(x, y) = N(x, y) = x^2+2xy. \tag{1}$$

Integrating the first relation with respect to x, we have

$$\varphi(x, y) = x^2y + xy^2 + x + h(y)$$

Differentiating, we have $\varphi_y = x^2+2xy+h'(y)$. Comparing with (1), we deduce that $h'(y) = 0$ and we may choose $h(y) = 0$. Hence, solutions are given by

$$\varphi(x, y) = x^2y + xy^2 + x = C.$$

And

$$C = \varphi(4, -2 + \sqrt{3}) = 16(-2 + \sqrt{3}) + 4(7 - 4\sqrt{3}) + 4) = 0.$$

Hence $0 = x^2y + xy^2 + x = x(xy + y^2 + 1)$. Since $x_0 = 4$, we deduce that $y^2 + xy + 1 = 0$. i.e.

$$y = \frac{-x \pm \sqrt{x^2 - 4}}{2}.$$

Using $y(4) = -2 + \sqrt{3}$ again, we deduce that

$$y = \frac{-x + \sqrt{x^2 - 4}}{2}, \qquad \text{for } x > 4.$$