Math 2255, Spring 2017, Quiz 4, Feb 27 **Name:** ____________________

1. [10] Find the solution of the given initial value problem, by the method of undetermined coefficients.

$$y'' + y' - 2y = 3e^t, \quad y(0) = 0, \quad y'(0) = 1.$$

Answer: First, the roots to the characteristic equation $r^2 + r - 2$ are $r_1 = -2$, $r_2 = 1$. So general solution to the homogeneous problem is

$$u_c(t) = c_1 e^{-2t} + c_2 e^t.$$

Next, we find a particular solution. Since $3e^t$ is a solution to the homogeneous problem, we let $Y(t) = Ate^t$. Then

$$Y' = Ae^t(1+t), \quad Y'' = Ae^t(2+t).$$

Hence

$$L[Y] = Y'' + Y' - 2Y = Ae^t[(2+t) + (1+t) - 2t] = 3Ae^t \Longrightarrow A = 1.$$

Then the general solution is given by

$$y(t) = u_c(t) + Y(t) = c_1 e^{-2t} + c_2 e^t + te^t$$

differentiating,

$$y'(t) = -2c_1 e^{-2t} + c_2 e^t + \frac{1}{3} e^t(1+t)$$

By the initial value $y(0) = 0$ and $y'(0) = 1$, we have

$$c_1 + c_2 = 0, \quad -2c_1 + c_2 + 1 = 1.$$

Hence we have $c_1 = 0$ and $c_2 = 0$, whence

$$y(t) = te^t.$$

2. [10] Use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y'' - 4ty' + 6y = 0, \quad t > 0, \quad y_1(t) = t^2.$$

Answer: Let $y(t) = v(t)y_1(t) = t^2v(t)$. Then

$$y' = 2tv + t^2v', \quad y'' = 2v + 4tv' + t^2v''.$$

Hence

$$\begin{aligned} t^2y'' - 4ty' + 6y &= t^2(2v + 4tv' + t^2v'') - 4t(2tv + t^2v') + 6t^2v \\ &= 4t^3v' + t^4v'' - 4t^3v' \\ &= t^4v''. \end{aligned}$$

Hence we need $v'' = 0$, i.e. $v(t) = c_1 + c_2t$. Therefore, we deduce $y(t) = (c_1 + c_2t)t^2 = c_1t^2 + c_2t^3$. i.e. the second equation is

$$y_2(t) = t^3.$$

3. [10] A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (where N stands for newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate (in S.I. units) the initial value problem describing the motion of the mass. [Hint: one newton N produces, for a mass of 1 kg, an acceleration of 1 ms^{-2}; the gravitational constant being g = 9.8 m s^{-2} and 1 m = 100 cm.]

Answer: The mass $m = 5\ kg$. By Hooke's law:

$$mg = kL \quad \Rightarrow \quad (5\ kg)(9.8\ ms^{-2}) = k(0.1\ m) \quad \Rightarrow \quad k = 490\ kg\ s^{-2}.$$

Next, to find the damping constant γ:

$$2\ N = 2\ ky\ m\ s^{-2} = \gamma(0.04ms^{-1}) \quad \Rightarrow \quad \gamma = 50\ kg\ s^{-1}.$$

Hence the problem can be formulated as

$$\begin{cases} 5u'' + 50u' + 490u = 10\sin\left(\frac{t}{2}\right), \\ u(0) = 0, \quad u'(0) = 0.04, \end{cases}$$

where the unit for u is m and the unit for t is s.