Math 2255, Spring 2017, Quiz 4, Mar 10 **Name:** ______________________________

1. [7] Find the inverse Laplace transform of $F(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$.

   Answer: Using the fact that $\mathfrak{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$, and that $\mathfrak{L}\{1\} = 1/s$, we have

$$\mathfrak{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\} = u_c(t)\cdot 1 = u_c(t),$$

   and hence

$$\begin{aligned}\mathfrak{L}^{-1}\{F(s)\} &= \mathfrak{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + \mathfrak{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathfrak{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \mathfrak{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\}\\ &= u_1(t) + u_2(t) - u_3(t) - u_4(t).\end{aligned}$$

2. [8] Find the Laplace transform of $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$.

   Answer: Write $f(t)$ in the form of $u_c(t)g(t-c)$....

$$f(t) = u_2(t)(t-2) - u_2(t) - u_3(t)(t-3) - u_3(t)$$

   Then

$$\begin{aligned}F(s) = \mathfrak{L}\{f(t)\} &= e^{-2s}\mathfrak{L}\{t\} - e^{-2s}/s - e^{-3s}\mathfrak{L}\{t\} - e^{-3s}/s\\ &= \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}\end{aligned}$$

3. [15] (a) Show that $\mathfrak{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$.

(b) Solve, using the Laplace transform method, the initial value problem

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0,\ y'(0) = 1.$$

Answer: (a) Using partial fraction: we want to write

$$\frac{1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2} = \frac{a(s+1)(s+2) + bs(s+2) + cs(s+1)}{s(s+1)(s+2)},$$

i.e. $1 = a(s+1)(s+2) + bs(s+2) + cs(s+1)$. By setting $s = 0, -1, -2$, we obtain $a = 1/2$, $b = -1$ and $c = 1/2$. Thus

$$\mathfrak{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \mathfrak{L}^{-1}\left\{\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Answer: (b) Taking Laplace Transform on both sides

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = e^{-2s}/s$$

i.e. $(s^2Y(s) - 1) + 3(sY(s)) + 2Y(s) = e^{-2s}/s$. Hence

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{e^{-2s}}{s(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} + e^{-2s}H(s),$$

where

$$H(s) = \frac{1}{s(s+1)(s+2)} \quad \text{satisfies by part (a)} \quad h(t) = \mathfrak{L}^{-1}\{H(s)\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Therefore,

$$\begin{aligned} y(t) = \mathfrak{L}^{-1}\{Y(s)\} &= e^{-t} - e^{-2t} + u_2(t)h(t-2) \\ &= e^{-t} - e^{-2t} + u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]. \end{aligned}$$