Math 2255, Spring 2017, Quiz 7, April 14 **Name:** ______________________

1. [15] Find the first 4 terms of the series solution about $x_0 = 0$ of

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad y(0) = 1, \; y'(0) = 2.$$

And determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $a_0 = 1$, $a_1 = 2$ and

$$(x^2 - 2x - 3)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - 2\sum_{n=1}^{\infty} a_{n+1}(n+1)n x^n - 3\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$
$$+ \sum_{n=1}^{\infty} a_n n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

$\boxed{n=0}$ $\; -3a_2 \cdot 2 + 4a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{2}{3}a_0 = \frac{2}{3}$

$\boxed{n=1}$ $\; -2a_2 \cdot 2 \cdot 1 - 3a_3 \cdot 3 \cdot 2 + a_1 + 4a_1 = 0$

$$\Rightarrow a_3 = \frac{-4a_2 + 5a_1}{18} = \frac{-4 \cdot \frac{2}{3} + 10}{18} = \frac{11}{27}$$

$$\therefore y(x) = 1 + 2x + \frac{2}{3}x^2 + \frac{11}{24}x^3 + \cdots$$

$x^2 - 2x - 3 = 0$ has roots $1 \pm \sqrt{2}$

$\Rightarrow y(x)$ converges at least in $(1-\sqrt{2}, \sqrt{2}-1)$

i.e. a lower bound of the radius of conv. is $\underline{\underline{\sqrt{2}-1}}$

See also next page for an alternative method:

Set $x = 0 \rightarrow -3y''(0) + 0 + 4y(0) = 0$

$$\Rightarrow \quad y''(0) = \frac{4}{3} y(0) = \frac{4}{3}.$$

Differentiate $\rightarrow (x^2-2x-3)y''' + (3x-2)y'' + 5y' = 0$

Set $x=0 \quad \Rightarrow -3y'''(0) - 2y''(0) + 5y'(0) = 0$

$$\Rightarrow y'''(0) = \frac{-2y''(0) + 5y'(0)}{3}$$

$$= \frac{-2\cdot\frac{4}{3} + 5\cdot 2}{3} = \frac{22}{9}$$

$$\Rightarrow \quad y(x) = 1 + 2x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \cdots$$

$$= 1 + 2x + \frac{2}{3}x^2 + \frac{11}{27}x^3 + \cdots$$

2. [5] Let $y = \phi(t)$ be the solution to

$$y'' + (\sin x)y' + (\cos x)y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Compute $\phi''(0)$, $\phi^{(3)}(0)$ and $\phi^{(4)}(0)$.

$$\phi'' + (\sin x)\phi' + (\cos x)\phi = 0$$

Set $x = 0$, $\boxed{\phi''(0) = 0 - (1\cdot)\phi(0) = 0}$

$$\phi''' + (\sin x)\phi'' + 2(\cos x)\phi' - \sin x\,\phi = 0$$

Set $x = 0$, $\phi'''(0) + 0 + 2\phi'(0) - 0 = 0$

$$\Rightarrow \boxed{\phi'''(0) = -2\phi'(0) = -2}$$

$$\phi^{(4)} + (\sin x)\phi''' + 3(\cos x)\phi'' - 3(\sin x)\phi' - \cos x\,\phi = 0$$

Set $x = 0$, $\phi^{(4)}(0) + 0 + 3\phi''(0) - 0 - \phi(0) = 0$

$$\phi^{(4)}(0) + 3\cdot 0 - 0 = 0 \Rightarrow \boxed{\phi^{(4)}(0) = 0}.$$

3. [10] Find γ so that solution of the initial value problem

$$x^2y'' - 2y = 0, \quad y(1) = 1, y'(1) = \gamma$$

remains bounded as $x \to 0$.

$x^2 y'' - 2y = 0$ is an Euler's equation

with indicial eq: $r(r-1) - 2 = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r = 2$ or -1

$\Rightarrow$ General sol is $y(x) = C_1 x^2 + C_2 x^{-1}$

$y'(x) = 2C_1 x - C_2 x^{-2}$

$$\begin{matrix} y(1)=1 \\ y'(1)=\gamma \end{matrix} \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ 2C_1 - C_2 = \gamma \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1+\gamma}{3} \\ C_2 = 1 - C_1 = \frac{2-\gamma}{3} \end{cases}$$

$$\Rightarrow y(x) = \left(\frac{1+\gamma}{3}\right) x^2 + \left(\frac{2-\gamma}{3}\right) x^{-1}.$$

$y(x)$ remains bounded as $x \to 0$ only if

$$\frac{2-\gamma}{3} = 0 \iff \boxed{\gamma = 2}$$