ON A PAPER BY EDMUNDS, KERMAN AND LANG

CHRISTER BENNEWITZ

1. INTRODUCTION

In [1] Edmunds, Kerman and Lang estimate the asymptotic behaviour of the approximation numbers of a simple operator $T$ on $L^2(a,b)$ defined by

$$T(f)(x) = v(x) \int_a^x u(t)f(t)\,dt, \quad x \in (a,b).$$

Here $u$ and $v$ are given real-valued functions in $C^1(a,b)$. The estimate obtained is

$$\limsup_{n \to \infty} n^{1/2} \frac{1}{\pi} \|uv\|_1 - na_n(T) \leq 3\sqrt{2} \left( \|u\|_{2/3}^2 + \|v\|_{2/3}^2 \right) \left( \|u\|_2 + \|v\|_2 \right) + \frac{3}{\pi} \|uv\|_1,$$

where $\| \cdot \|_p$ denotes the $L^p$ (quasi-)norm on $(a,b)$. We shall show that the approximation numbers are given by $a_n(T) = \lambda_n^{-1/2}$, where $\lambda_n, n = 1,2,\ldots$ are the eigen-values of an associated Sturm-Liouville operator. Thus any information on these eigen-values gives corresponding information on the approximation numbers. In particular, we will show that the left hand side in (1.1) often is 0; in fact, under some conditions on $u$ and $v$ we will find a finite value for $\lim_{n \to \infty} n(a_n(T) - \frac{1}{\pi} \|uv\|_1)$.

2. REDUCTION TO A STURM-LIOUVILLE EQUATION

The definition of approximation numbers, in the case of an operator $T$ between Hilbert spaces, means that they are the singular values of $T$, i.e., the square roots of the eigenvalues of $T^*T$. A simple calculation shows that $T^*(f)(x) = \bar{u}(x) \int_x^b \bar{v}(t)f(t)\,dt$ so that we are looking at the operator

$$T^*T(f)(x) = \bar{u}(x) \int_x^b |v(t)|^2 \int_a^t u(s)f(s)\,ds\,dt.$$ 

Suppose $f$ is an eigen-function, so that $T^*Tf = a^2f$, where $a$ is the corresponding approximation number of $T$. Setting $g = T^*Tf/\bar{u} = a^2f/\bar{u}$ we then have $-\left( |v|^{-2}g' \right)' = uf = a^{-2}|u|^2g$ and $g(b) = |v(a)|^{-2}g'(a) = 0$. 

1
so that $a^{-2}$ is an eigen-value to the Sturm-Liouville problem

\begin{equation}
\begin{cases}
-(|v|^{-2}g)' = \lambda |u|^2 g, \\
|v(a)|^{-2}g'(a) = g(b) = 0,
\end{cases}
\end{equation}

considered in the $L^2$-space with weight $|u|^2$ on the interval $[a, b]$. It is equally easy to show that, conversely, every such eigen-value is the inverse square of an approximation number for $T$. Any information on the distribution of eigen-values for the Sturm-Liouville problem thus implies similar information on the approximation numbers of $T$. There is a very large literature on such eigen-value estimates; we will only discuss one such result, deduced under similar, but not identical, assumptions compared to those of [1]. To do this it is convenient to make a change of independent variable in (2.1). Thus we set $t = \int_0^{\pi} |uv|$. This maps the interval $[a, b]$ onto $[0, A]$, where $A = ||uv||$. Our Sturm-Liouville problem is transformed into

\begin{equation}
\begin{cases}
-(pf')' = \lambda pf \text{ on } [0, A], \\
pf'(0) = f(A) = 0,
\end{cases}
\end{equation}

where $p(t(x)) = |u(x)|/|v(x)|$. We assume that $u$ and $v$ are absolutely continuous, possibly complex-valued, functions which vanish at most on a nullset. Now $p(t) \, dt = |u(x)|^2 \, dx$ and $\frac{dt}{p(t)} = |v(x)|^2 \, dx$, so since $u$ and $v \in L^2(a, b)$, it follows that $p$ and $1/p$ are in $L^1(0, A)$. We shall also need that $p'/p$ is integrable, and since $p'(t)/p(t) \, dt = (|u'|/|u| - |v'|/|v|) \, dx$ we will therefore assume that the logarithmic derivative of $|u/v|$ is integrable. This amounts to assuming that $|u/v|$ is finite and non-zero in $(a, b)$, with finite and non-zero limits at the endpoints.

### 3. Eigenvalue estimates

We consider a Sturm-Liouville equation $-(pu')' = \lambda pu$ on a compact interval $[0, A]$ with boundary conditions $u(0) = pu'(A) = 0$. We assume $p \geq 0$ and that $p$ and $1/p$ are locally integrable, and consider the problem in the weighted $L^2$-space on $(0, A)$ with weight $p$. The spectrum is then discrete, and to investigate the eigen-values we make a Prüfer transformation

\begin{align*}
f &= r \cos \theta, \\
f' &= -\sqrt{\lambda} \, r \sin \theta.
\end{align*}

It is easily seen that $-(pf')' = \lambda pf$ then implies the equation

\begin{equation}
\theta' = \sqrt{\lambda} - \frac{p'}{2p} \sin(2\theta).
\end{equation}
The boundary condition at 0 gives \( \sin \theta(0) = 0 \), so we may assume \( \theta(0) = 0 \). Thus

\[
\theta(t) = t\sqrt{\lambda_n} - \int_0^t \frac{p'}{2p} \sin(2\theta),
\]

so we assume that \( p'/p \) is integrable. The boundary condition at \( A \) gives \( \cos \theta(A) = 0 \), and by the Sturm oscillation theorem the \( n \):th eigen-function has \( n - 1 \) interior zeros. Since \( \theta(0) = 0 \) this implies that if \( \lambda = \lambda_n \) is the \( n \):th eigen-value, then \( \theta(A) = (2n - 1)\pi/2 \). We therefore obtain

\[
\frac{(2n - 1)\pi}{2} = A\sqrt{\lambda_n} - \int_0^A \frac{p'}{2p} \sin(2\theta).
\]

Since \( a_n = \lambda_n^{-1/2} \) and \( A = \|uv\|_1 \) we obtain

\[
a_n = \frac{2\|uv\|_1}{(2n - 1)\pi} \left( 1 + \frac{2}{(2n - 1)\pi} \int_0^A \frac{p'}{2p} \sin(2\theta) \right)^{-1}.
\]

We shall show presently that the integral tends to 0 as \( n \to \infty \), and then it immediately follows that

\[
(3.1) \quad a_n = \frac{2}{(2n - 1)\pi} \|uv\|_1 (1 + o(1/n)).
\]

In particular this yields

\[
\lim n(na_n - \frac{1}{\pi} \|uv\|_1) = \frac{1}{2\pi} \|uv\|_1.
\]

as claimed in the introduction. However, if \( |u| = |v| \), then \( p \) is constant, so in this case \( a_n = \frac{2}{(2n - 1)\pi} \|uv\|_1 \). It therefore seems more appropriate to write the formula on the form (3.1).

To show that the integral tends to 0 we set

\[
I(t) = \lim \sup_{n \to \infty} \left| \int_0^t \frac{p'}{2p} \sin(2\theta) \right|.
\]

It is of course clear that the integral has a bound independent of \( n \), and by Riemann-Lebesgues lemma \( \int_0^t \frac{p'}{2p} \sin(2s\sqrt{\lambda_n}) \, ds \to 0 \) as \( n \to \infty \). Furthermore \( \left| \sin(2\theta(t)) - \sin(2t\sqrt{\lambda_n}) \right| \leq 2 \left| \int_0^t \frac{p'}{2p} \sin(2\theta) \right| \). From this follows that \( I(t) \leq \int_0^t \frac{p'}{|p|} I(s) \, ds \). Thus \( I \equiv 0 \) by Gronwalls lemma.

Finally, note that

\[
\left| \int_0^A \frac{p'}{2p} \sin(2\theta) \right| = \frac{1}{2} \left| \int_a^b (|u'|/|u| - |v'|/|v|) \sin(2\theta) \right| \leq \int_a^b ||u'|/|u| - |v'|/|v||/2,
\]

so we also have a bound on the integral valid for all \( n \).

If we only assume local integrability of \( p'/p \) we can still obtain estimates, if we assume that \( t^np'(t)/p(t) \) is integrable near \( t = 0 \),
for some $\alpha$, $0 < \alpha \leq 1$, and a similar condition at $A$. Roughly, this is because $\sin(2\theta)$ may be bounded by $\min(1, Cn, Cn(A-t)) \leq \min(1, (Cn)^2, (Cn(A-t))^2)$ with an appropriate constant $C$. Thus, to estimate the integral we split the interval in 3 pieces; $[0, \delta]$, $[\delta, A-\delta]$ and $[A-\delta, A]$. The middle part tends to 0 as before, and in $[0, \delta]$ we replace $\sin(2\theta)$ by $(Cn)^\alpha$, so this part is estimated by $Bn^\alpha$, where $B$ is a constant depending on $\delta$, which becomes arbitrarily small with $\delta$. A similar estimate is made for the third part, the result being that the integral is $o(n^\alpha)$. Thus we obtain $a_n = \frac{2||w||_1}{(2n-1)\pi} (1 + o(n^{\alpha-1}))$.

REFERENCES


DEPARTMENT OF MATHEMATICS, LUND UNIVERSITY, BOX 118. SE-221 00 LUND, SWEDEN

E-mail address: christer.bennewitz@math.lu.se