The authors study a number of problems arising from the question of what odd real functions $f(x)$ of $L^2(-\pi, \pi)$ can satisfy

$$\pi^{-1} \int_{-\pi}^{\pi} f(nx) f(mx) \, dx = \delta_{mn}, \quad n, m = 1, 2, \ldots$$

The only complete set $\{f(nx)\}$ satisfying (1), or even the weaker condition with $(n, m) \leq 1$, arises from $f(x) = \pm \sin x$; for any other $f(x)$, the set $\{f(nx)\}$ cannot even be completed by adjoining a finite number of functions. Sharper results of similar character are given. Let $f(x) \sim \sum_{k=1}^{\infty} a_k \sin kx$. Then the condition that $f(x)$ satisfies (1) and $\sum |a_k| < \infty$ is equivalent to the following condition on $\varphi(z) = \sum a_k k^{-z}$: $\varphi(z)$ is meromorphic, the Dirichlet series defining $\varphi(z)$ converges absolutely for $\Re(z) \geq 0$, and $\varphi(z)\varphi(-z) = 1$. Using this equivalence, the authors obtain solutions of (1) and discuss the construction of further solutions by combining them. They also discuss transformations of classes of solutions of (1); a wide variety of special conditions which force solutions to have special forms; topological properties in $l^2$ of the sequences $\{a_k\}$ corresponding to solutions of (1); and a theorem on singularities of power series associated with the problem.

Reviewed by R. P. Boas, Jr.