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On the solution of some nonlinear boundary value problem

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Abstract. The subject of this paper is to give classical solutions for the Dirichlet and Neumann problem to the quasilinear equation of the divergence form

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^p + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^p + \lambda u^p = 0 \quad \text{in} \quad \Omega \in \mathbb{R}^2,$$

where \( p > 0 \) real, the function \( v^p \) is defined by

$$v^p = \begin{cases} v^p & \text{if } v \geq 0, \\ -|v|^p & \text{if } v < 0, \end{cases}$$

and \( \Omega \) is bounded by a rectangle. We give the eigenfunctions and the corresponding eigenvalues moreover an asymptotic formula for the eigenvalues in both cases when the Dirichlet and when the Neumann condition are satisfied.

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0. Introduction

The knowledge of the vibrational characteristics is of considerable importance for the designer of safe and efficient structures. Such structures may appear in microphones, bridges, ships or in space vehicles. It is therefore essential to employ an analysis to predict static and stability behaviour of such elements of structures.

For the determination of the motion of a supported membrane one has to solve the partial differential equation

$$\Delta u + \lambda u = 0 \quad (0.1)$$

in the domain \( \Omega \) with the boundary condition. The Dirichlet problem yields the solution \( u \in L^2(\Omega) \) of the differential equation (0.1) with condition

$$u|_{\partial \Omega} = 0. \quad (0.2)$$

The Neumann problem yields the solution \( u \in L^2(\Omega) \) of (0.1) with condition

$$\frac{\partial u}{\partial n}|_{\partial \Omega} = 0. \quad (0.3)$$
Here $\frac{\partial}{\partial n}$ denotes the outward normal derivative. A considerable number of studies have been performed for rectangular and circular geometries where the determination of natural frequencies and mode shapes is relatively easy [1-3], [5-7], since the differential equation (0.1) can be transformed into separable form where ordinary differential equations have to be solved. Eisenhart showed that for circle, ellipse, and circular, elliptic, and parabolic segment the variables can be separated in the differential equation (0.1) when $\Omega \in R^2$.

Let $N(\Lambda) = \sum_{\lambda_i \leq \Lambda} 1$, and $\lambda_i > 0$. For sufficiently large $\Lambda$ the asymptotic formula of the eigenvalue problem (0.1) and (0.2) or (0.3) has the form

$$\lim_{\Lambda \to \infty} \frac{N(\Lambda)}{\Lambda} = \frac{\text{mes} \Omega}{4\pi}$$

where $\text{mes} \Omega$ means the area of the domain $\Omega$. In 1912 Weyl made a following conjecture

$$N(\Lambda) = \frac{\text{mes} \Omega}{4\pi} \Lambda \pm \frac{\text{mes} \partial \Omega}{4\pi} \sqrt{\Lambda} + O(\sqrt{\Lambda}),$$

where $\text{mes} \partial \Omega$ is the length of the boundary $\partial \Omega$, and the positive sign is concerned with the Neumann condition, the negative sign with the Dirichlet boundary condition. Kuznecov proved formula (0.4) for eigenvalues of a circular membrane [9] and of an elliptic membrane [10].

1. Preliminaries

We consider the quasilinear equation of divergence form in two variables

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^p + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^p + \lambda u^p = 0 \quad \text{in} \quad \Omega \quad (1.1)$$

where $p > 0$ real and the function $v^p$ is defined by

$$v^p = \begin{cases} v^p & \text{if } v \geq 0, \\ -|v|^p & \text{if } v < 0. \end{cases}$$

Let $\Omega \in R^2$ be a bounded domain with smooth boundary $\partial \Omega$. The boundary condition corresponding to the Dirichlet problem is

$$u|_{\partial \Omega} = 0 \quad (1.2)$$

and the eigenvalues and eigenfunctions are denoted $\lambda^{(1)}$ and $u^{(1)}$, respectively. The boundary condition of the Neumann problem is

$$\frac{\partial u}{\partial n}|_{\partial \Omega} = 0, \quad (1.3)$$
where \( \frac{\partial}{\partial n} \) denotes the outward normal derivative, \( \lambda^{(2)} \) and \( u^{(2)} \) denote the eigenvalues and eigenfunctions, respectively.

It was proved in [4] that the Dirichlet problem of (1.1) for \( 1 \leq p < \infty \) has at least one nontrivial nonnegative weak solution \( u \in W^{1,p+1}_0(\Omega) \) in \( \mathbb{R}^n \). We shall show that the Dirichlet and Neumann problem of (1.1) have solutions belonging to \( C^2(\Omega) \cap C^1(\Omega) \) when \( \Omega \) is bounded by the rectangle

\[ \Omega = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}. \tag{1.4} \]

2. The eigenvalue problem for a rectangle

We consider the eigenvalue problem of (1.1) under condition (1.2) or (1.3) in the rectangle given by (1.4).

**Theorem 1.1** For the Dirichlet problem of (1.1)

\[
\lambda_{k,l}^{(1)} = p\pi^{p+1} \left( \frac{k^{p+1}}{ap^{1}} + \frac{l^{p+1}}{bp^{1}} \right), \tag{2.1}
\]

\[ u_{k,l}^{(1)} = A_{k,l} S_p \left( \frac{k\pi}{a} x \right) S_p \left( \frac{l\pi}{b} y \right), \quad k, l = 1, 2, \ldots \]

are eigenvalues and eigenfunctions, respectively and for the Neumann problem of (1.1)

\[
\lambda_{k,l}^{(2)} = p\pi^{p+1} \left( \frac{k^{p+1}}{ap^{1}} + \frac{l^{p+1}}{bp^{1}} \right), \tag{2.2}
\]

\[ u_{k,l}^{(2)} = B_{k,l} S_p \left( \frac{k\pi}{a} x + \frac{\pi}{2} \right) S_p \left( \frac{l\pi}{b} y + \frac{\pi}{2} \right), \quad k, l = 0, 1, 2, \ldots \]

are eigenvalues and eigenfunctions respectively, \( \Omega \) is bounded by the rectangle \( \Omega = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\} \) and \( \pi = \frac{2\pi}{p+1} \sin \frac{\pi}{p+1} \), \( A_{k,l} = \text{const} \), \( B_{k,l} = \text{const} \), and the function \( S_p(x) \) is the solution of the differential equation

\[ S''_p |S'_p|^{p-1} + S''_p = 0 \]

under condition \( S_p(0) = S_p(\pi) = 0 \).

**Conjecture.** If the domain \( \Omega \) is rectangular then all the solutions of the Dirichlet and Neumann problem of (1.1) has the multiplicative form \( u(x,y) = r(x)q(y) \).

**Proof.** We look for the solution \( u^{(1)} \) of the eigenvalue problem (1.1), (1.2) in the form

\[ u(x,y) = r(x)q(y). \tag{2.3} \]
Substituting (1.4) to the equation (1.1) we get

\[ \left[ (r')^p \right]' q^p + r^p \left[ (q')^p \right]' = 0 \]  

(2.4)

where \( x \in (0, a), y \in (0, b) \). We suppose that \( u^{(1)}(x, y) = r(x)q(y) \neq 0 \) at some point \((x, y)\). From the equation (2.4) we have

\[ \frac{p |r'|^{p-1}r''}{r^p} = -\lambda - \frac{p |q'|^{p-1}q''}{q^p}. \]

For the functions \( r(x) \) and \( q(y) \) we get the following half-linear second order ordinary differential equations

\[ r'' |r'|^{p-1} + \frac{\mu}{p} r^p = 0 \quad \text{if} \quad x \in (0, a), \]

(2.5)

\[ q'' |q'|^{p-1} - \frac{\nu}{p} q^p = 0 \quad \text{if} \quad y \in (0, b), \]

(2.6)

and \( \lambda = \mu + \nu \). Now the boundary condition (1.2) can be written as

\[ r(0) = r(a) = 0, \]

(2.7)

\[ q(0) = q(b) = 0. \]

(2.8)

A solution of the problem (2.5), (2.7) is given by A. Elbert [8] as \( r(x) = S_p(x) \) in the case \( \mu = p \) where

\[ x = \int_0^s \frac{d\sigma}{r^p \sqrt{1 - \sigma^{p+1}}} \quad \text{in} \quad [0, \frac{\pi}{2}] \]

and

\[ \tilde{\pi} = \frac{2\pi}{p + 1} \sin \frac{\pi}{p + 1}. \]

(2.9)

\( S_p(x) \) can be extended to the whole real axis:

\[ S_p(x) = \begin{cases} 
S_p(\tilde{\pi} - x) & \text{if} \quad \frac{\pi}{2} \leq x < \tilde{\pi}, \\
-S_p(x - \tilde{\pi}) & \text{if} \quad \tilde{\pi} \leq x < 2\tilde{\pi}, \\
S_p(x + 2k\tilde{\pi}) & \text{where} \quad k = \pm 1, \pm 2, ... 
\end{cases} \]

(2.10)

Furthermore we have that

\[ S(x) = 0 \quad \text{if} \quad x = 0, \pm\tilde{\pi}, \pm 2\tilde{\pi}, ... \]

\[ S'(x) = 0 \quad \text{if} \quad x = 0, \pm\frac{\tilde{\pi}}{2}, \pm 3\frac{\tilde{\pi}}{2}, ... \]

....
The function $S_p(x)$ plays the same role in the case of halflinear differential equations as $\sin x$ in the case of linear differential equations. We have the relation
\[ |S_p(x)|^{p+1} + |S_p'(x)|^{p+1} = 1 \quad x \in \mathbb{R} \]
which can be considered as a generalization of the Pythagorean relation.

In the case $\mu \neq p$ by substitution of $r(x) = S_p(cx)$ we get
\[ S_p''|S_p'|^{p-1} + \frac{\mu}{pc^{p+1}} S_p^* = 0, \quad S_p(0) = S_p(a) = 0 \quad (2.10) \]
and necessarily
\[ \mu = pc^{p+1}. \]

Satisfying the boundary condition we find
\[ c = \frac{k\pi}{a} \quad (k \in \mathbb{N}) \quad \text{and} \quad \mu = p\frac{k^{p+1}a^{p+1}}{a^{p+1}}. \]

Apart from a constant factor the function $r(x)$ can be only one of the functions
\[ r_k(x) = S_p\left(\frac{k\pi}{a} x\right). \]

The solution of the problem (2.6), (2.8) could be found analogously
\[ g_l(y) = S_p\left(\frac{l\pi}{b} y\right) \quad (l \in \mathbb{N}), \quad \text{and} \quad \nu = p\frac{l^{p+1}a^{p+1}}{b^{p+1}}. \]

The eigenvalue $\lambda_{k,l}^{(1)}$ is obtained by $\lambda = \mu + \nu$.

The solution of the eigenvalue problem (1.1), (1.3) can be determined in the same way. Thus we have the eigenfunctions $u_{k,l}^{(2)}(x,y)$ and the corresponding eigenvalues $\lambda_{k,l}^{(2)}$. We remark that also the values $k = 0, l = 0$ are admitted in (2.2) while in the case of Dirichlet problem no.

All the solutions of the Dirichlet and Neumann problem, which are of product form (2.3), has been found. We know that in the linear case $p = 1$ the functions \( \{u_{k,l}\}_{k=1,l=1}^{\infty} \) form a complete system in $L^2(0,\pi)$. The same is not known for the nonlinear case but we hope so.

3. The asymptotic distribution of eigenvalues for a rectangle

For fixed domain $\Omega$ we have seen the infinite series of $\lambda_{k,l}^{(i)} \quad (i = 1, 2)$. For the investigation of the asymptotic distribution of eigenvalues is suitable to define the function
\[ N^{(i)}(\Lambda) = \sum_{\lambda_{k,l}^{(i)} > \Lambda} 1, \quad i = 1, 2. \]
In the linear case \((p=1)\) Weyl's conjecture is known (0.4). For arbitrary \(p(>0)\) we give a similar connection under validity of our conjecture when \(\Omega\) is bounded by a rectangle.

**Theorem 3.1** For the asymptotic distribution of the corresponding eigenvalues (2.1) and (2.2) to the Dirichlet and Neumann problem the following connection holds

\[
N^{(i)}(\Lambda) = \frac{abB\left(\frac{1}{p+1}, \frac{1}{p+1}\right)}{\pi^2(p+1)^{p+1}} \frac{r_{+}\Lambda^2}{\sqrt{\Lambda}} \pm \frac{a + b}{2\pi r_{+}\sqrt{\Lambda}} \frac{r_{+}\sqrt{\Lambda}}{r_{+}^{p+1}} O^{1-\frac{1}{p+1}} \left(\frac{r_{+}\sqrt{\Lambda}}{r_{+}^{p+1}}\right),
\]

where the \((+)\) sign is concerned with \(i=2\) and the \((-)\) sign with \(i=1\).

For the proof of this theorem we shall apply the next two lemmas:

**Lemma 3.2** (see [7 p. 430]) Let \(f(x) > 0\) be strictly increasing function on the interval \(a \leq x \leq b\) and let us denote the domain \(D = \{(x,y) : a \leq x \leq b, f(a) \leq y \leq f(x)\}\), the area of \(D\) by \(A(D)\) and the number of lattice points contained in \(D\) by \(I(D)\). Then the inequality

\[
|A(D) - I(D)| \leq |f(b) - f(a)| + |b - a| + 1
\]

is valid where \([x]\) is the integer part of \(x\).

**Lemma 3.3** (van der Corput [12 p. 279]) Let \(s < w, \text{and } s - \frac{1}{2}, w - \frac{1}{2}, q - \frac{1}{2}\) be integers. Let \(f(x)\) be differentiable as \(s \leq x \leq w, f(s) > q+1, \text{moreover } f'(x) > 0, f(u) \text{ be either convex } (f'(u) \text{ is increasing}) \text{ or concave } (f'(u) \text{ is decreasing}) \text{ and}

\[
0 < m = \min f'(x) \leq f'(x) \leq \max f'(u) = M.
\]

Let \(r > 1\) such that

\[
\frac{x_2 - x_1}{|f'(x_2) - f''(x_1)|} < r \quad \text{if} \quad s \leq x_1 < x_2 \leq w
\]

(if \(f''(x)\) exists then \(\frac{1}{|f'(x)|}\) is bounded by \(r\)) and \(r > \frac{1}{m^3}\).

Denote the curvilinear trapezoid \(s \leq x \leq w, q \leq y \leq f(x)\) by \(D\) and let \(A(D)\) and \(I(D)\) be defined in Lemma 3.2. Then

\[
|A(D) - I(D)| \leq cr^3 M \quad (c = \text{const.}).
\]

**Remark 3.4** The statement of Lemma 3.3 can be also generalized in the case \(f'(x) < 0\).

**Proof of Theorem 2.1.** Let us consider the domain \(D\) in the plane \((\xi, \eta)\) defined by

\[
\Lambda < p \pi^{p+1} \left(\frac{\xi^{p+1}}{a^{p+1}} + \frac{\eta^{p+1}}{b^{p+1}}\right), \quad \xi > 0, \eta > 0.
\]
The area of the domain $D$ is

$$A(D) = b \int_0^{p+1} \sqrt{\frac{\Lambda}{p}} \left( 1 + \frac{\xi^{p+1}}{a^{p+1}} \right) d\xi = \frac{abB\left(\frac{1}{p+1}, \frac{1}{p+1}\right)}{\pi^2(p+1)^{p+\sqrt{\Lambda}/2}}.$$  \hspace{1cm} (3.3)

First we consider the eigenvalues of the Dirichlet problem $\lambda_{k,l}^{(1)}$. By (2.1) and (3.2) the number $N(1)(\Lambda)$ of the eigenvalues $\lambda_{k,l}^{(1)}$ less than $\Lambda$ is equal to the number of the pairs $(k,l)$ ($k,l \in N$) contained in $D$. A rough estimation for $N(1)(\Lambda)$ we can derive by Lemma 3.2:

$$|A(D) - N(1)(\Lambda)| \leq \left[ \frac{a}{\pi} \frac{p+1}{\sqrt{p}} \right] + \left[ \frac{b}{\pi} \frac{p+1}{\sqrt{p}} \right] + 1 < \frac{a+b}{\pi^{p+\sqrt{\Lambda}}}.$$  \hspace{1cm} (3.1)

On comparing the right-hand side here with (3.1) we find that the error is of the same magnitude as the second term of the expression (3.1). To prove (3.1) we have to follow a more sophisticated way. Let the function

$$f(\xi) = \frac{b}{a} \frac{p+1}{L^{p+1} - \xi^{p+1}} = \frac{\Lambda}{p^{p+1}}, \quad 0 < \xi < L,$$

be differentiable and

$$f'(\xi) = \frac{b}{a} \left( \frac{\xi^{p+1}}{L^{p+1} - \xi^{p+1}} \right)^{p+1},$$

$$f''(\xi) = -\frac{b}{a} \frac{L^{p+1} \xi^{p-1}}{(L^{p+1} - \xi^{p+1})^{p+1}},$$

so $f'(0) = 0$, $\lim_{\xi \to L-0} f'(\xi) = -\infty$, $f''(\xi) < 0$. There is unique $\xi_0 \in (0,L)$ such that

$$f'(\xi_0) = -1$$

hence

$$\xi_0 = L \frac{\left(\frac{1}{b}\right)^{p+1}}{1 + \left(\frac{p+1}{p+1}\right)^{p+1}}.$$  \hspace{1cm} (3.2)

Let us consider the domain $D' = D \cap H$,

$$H = \{ (\xi, \eta) : \xi \geq \frac{1}{2}, \eta \geq \frac{1}{2} \}.$$
Clearly

\[ A(D) - A(D') = \frac{a + b}{2\pi} r^{+1} \sqrt{\frac{\Lambda}{p}} + O(1). \]  

(3.4)

Let us observe that by (2.1)

\[ N^{(1)}(\Lambda) = I(D') \]  

(3.5)

and by (2.2)

\[ N^{(2)}(\Lambda) = I(D') + \left( \frac{a}{\pi} r^{+1} \sqrt{\frac{\Lambda}{p}} + \frac{b}{\pi} r^{+1} \sqrt{\frac{\Lambda}{p}} \right) - 1. \]  

(3.6)

Let us divide the domain \( D' \) into five parts

\[ D_0 = \{(\xi, \eta) : \left( \frac{1}{2} \leq \xi \leq s - 1, \quad \frac{1}{2} \leq \eta \leq t - 1 \} \}, \]

\[ D_{01} = \{(\xi, \eta) : (s - 1 \leq \xi \leq s, \quad \frac{1}{2} \leq \eta < \min \{t, f(\xi) \} \}, \]

\[ D_{02} = \{(\xi, \eta) : \left( \frac{1}{2} \leq \xi \leq s - 1, \quad t - 1 \leq \eta \leq t \} \}, \]

\[ D_1 = \{(\xi, \eta) : (s \leq \xi < L, \quad \frac{1}{2} \leq \eta < f(\xi) \}, \]

where \( s = [2\xi_0] - [\xi_0] + \frac{1}{4}, \quad t = [2f(\xi_0)] - [f(\xi_0)] + \frac{1}{4}. \)

In the rectangles \( D_0, D_{02} \) the numbers of lattice points \( I(\cdot) \) are equal to the corresponding areas \( A(\cdot) \). In the domain \( D_{01} \) the difference between \( A(D_{01}) \) and \( I(D_{01}) \) is less than 1. The domain \( D_1 \) will be divided into two parts

\[ D_{11} = \{(\xi, \eta) : (s \leq \xi \leq w, \quad \frac{1}{2} \leq \eta < f(\xi) \}, \]

\[ D_{12} = \{(\xi, \eta) : (w \leq \xi < L, \quad \frac{1}{2} \leq \eta < f(\xi) \}, \]

where \( w = [L - L^{3/4}] + \frac{1}{2}. \)

In both cases we have to find an upper bound for \( |A(\cdot) - I(\cdot)|. \) We mention that \( w > s \) for sufficiently large \( L \) or \( \Lambda \). Let us consider the curvilinear trapezoid \( D_{11} \). Each condition of Lemma 3.3 are satisfied with \( q = \frac{1}{2} \). The point \( (w, \frac{3}{4}) \) is in \( D_1 \) which is equivalent to \( f(w) > q + 1 \) if \( \Lambda \) is sufficiently large. For the derivatives we have the estimates

\[ m = \min |f'(\xi)| = |f'(s)| > |f'(\xi_0)| = 1, \]

\[ M = \max |f'(\xi)| = |f'(w)| \leq C_1(a, b, p) L^{3/4} p^{1/4} + 1, \]
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\[ \min |f''(\xi)| = |f''(s)| \geq |f''(\xi_0)| = C_2(a, b, p) \frac{1}{L}, \]

where \( C_i, (i = 1, 2) \) denotes a constant independent of \( L \). Hence for \( r \) in Lemma 3.3 we have

\[ r > \frac{1}{C_2 L} \]

and applying Lemma 3.3 we obtain

\[ |A(D_{11}) - I(D_{11})| \leq \text{const}.L^{-1 - \frac{1}{3p + 1}}. \]

For \( D_{12} \) we have by Lemma 3.2

\[ |A(D_{12}) - I(D_{12})| = |L - w| + |f(w)| + 1 \leq \text{const}.L^{-1 - \frac{1}{3p + 1}} \]

when \( \Lambda \) is sufficiently large. Therefore in \( D_1 \) we get

\[ |A(D_1) - I(D_1)| \leq \text{const}.L^{1 - \frac{1}{3p + 1}} = \text{const}. \left( \frac{\Lambda}{p} \right)^{1 - \frac{1}{3p + 1}}. \]

By the same way we have a similar inequality for \( D_2 \). Therefore

\[ |A(D') - I(D')| \leq \text{const}. \left( \frac{\Lambda}{p} \right)^{1 - \frac{1}{3p + 1}}. \]

when \( \Lambda \) is sufficiently large. Therefore from (3.3-7) we have (3.1).

References