EIGENVALUE PROBLEM FOR A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATION

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We consider the boundary value problem of the nonlinear differential equation \[
\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[ \frac{\partial}{\partial x_i} \right]^p u + \lambda u = 0 \quad (0 < p < \infty) \] in \( D \in \mathbb{R}^N \) under the Dirichlet and Neumann boundary condition. The classical solution for different domains \( D \) will be given when \( D \) is bounded by a rectangle or by a central symmetric convex curve.

Keywords and Phrases: Nonlinear systems; partial differential equations; eigenvalue problems; boundary value problem; radially symmetric solutions

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1. INTRODUCTION

The leading example of a linear eigenvalue problem is to find all nontrivial solutions of the equation

\[
\Delta u + \lambda u = 0 \quad \text{in} \; D
\] (1)

in the bounded domain \( D \in \mathbb{R}^N \) under various boundary conditions. We can consider on the boundary of \( D \), \( \partial D \) either the Dirichlet condition

\[
u = 0 \quad \text{on} \; \partial D,
\] (2)

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or the Neumann condition

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.
\]

(3)

where the symbol \( \partial u/\partial n \) denotes the directional derivative of \( u \) along the outward normal to the boundary of \( D \). It is known that the linear eigenvalue problem (1)–(2) or (1)–(3) in bounded domain has eigen-solutions \( u_j \in L^2(D) \) and corresponding eigenvalues \( \lambda_j \ (j = 1, 2, \ldots) \) such that \( \lambda_j \to \infty \) when \( j \to \infty \). These problems are solved by several methods in some special domains.

The assumption of linearity underlies, as a fundamental postulate, a considerable domain of mathematics. However, nature, with scant regard for the desires of the mathematician, is essentially nonlinear – that is to say, the mathematical models believed to best approximate her (they are of course all approximations) are nonlinear.

Needless to say, that the linear equation (I) has been generalized in numerous ways: to Riemann surfaces and manifolds, to equation \( \Delta u + \lambda u + Vu = 0 \) with a potential \( V \), to more general differential operators than the Laplacian, and so on.

However, when talking about nonlinear eigenvalue problems, there is seldom any eigenvalue at all involved.

For example, one just considers the existence of positive solutions. The extremely popular and very interesting Emden-Fowler equation

\[
\Delta u - \lambda u|u|^\alpha = 0
\]

is of this type.

We can consider as a generalization of the linear problem (1)–(2) the following nonlinear eigenvalue problem.

\[
\sum_{i=1}^{N} \frac{\partial}{\partial x_i} (|\nabla \hat{u}|^p) + \lambda \hat{u}^p = 0
\]

(4)

with \( u = 0 \) on the boundary of a bounded domain \( D \) in the \( N \)-dimensional Euclidean space. Here \( 0 < p < \infty \) and the function \( \hat{u}^p \) is defined as follows:

\[
\hat{u}^p = |u|^{p-1} u = \begin{cases} 
  u^p & \text{if } u > 0, \\
  -|u|^p & \text{if } u \leq 0.
\end{cases}
\]
Note that for $p = 1$ we are back to the linear case $\Delta u + \lambda u = 0$. If $u$ is a solution, not identically zero, the value $\lambda$ can be expressed by

$$\lambda = \frac{\int_D |\nabla u|^{p+1} \, dx}{\int_D |u|^{p+1} \, dx}.$$  

Thus it appears that $\lambda > 0$. Minimizing this so called nonlinear Rayleigh quotient among all admissible functions we arrive at (4) as the corresponding Euler-Lagrange equation. The first one to study it seems to have been F. de Thelin [5] in 1984. The so called $p$-harmonic operator or $p$-Laplacian $\sum_{i=1}^N \frac{\partial}{\partial x_i} (|\nabla u|^p)\frac{\partial u}{\partial x_i}$ appears in many contexts in physics: non-Newtonian fluids, reaction-diffusion problems, nonlinear elasticity, and glaciology, just to mention a few applications.

Another generalization of (1) is the following equation

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left[ \frac{\partial u}{\partial x_i} \right]^{p-1} \frac{\partial^p u}{\partial x_i} - \lambda u^p = 0 \quad \text{in } D \subset R^N. \quad (5)$$

If $p = 1$ then the nonlinear equation (5) also gives the linear equation (1).

Here our main purpose is to give results on the solution of the Dirichlet and Neumann problem of the nonlinear partial differential equation (5).

2. PRELIMINARY RESULTS

Let $F$ be defined as the set of real functions $v(x) : v(x) \in W^{1,p+1}_0(D)$, $v|_{\partial D} = 0$ and the quotient $R[v]$—called generalized Rayleigh quotient,—be defined as follows:

$$R[v] = \frac{\int_D \sum_{i=1}^N |\partial v_i|^{p+1} \, dx}{\int_D |v|^{p+1} \, dx}, \quad v \in F$$

The Euler-Lagrange equation is the Eq. (5), which corresponds to the variational problem of minimizing the Rayleigh quotient. Since $R[v] = R[cv]$ ($c$ = constant), then the solutions can be made unique by the introduction of a suitable norm. As the linear case if there exist constants $\lambda_j$ and corresponding functions $u_j$ ($j = 1, 2, \ldots$) satisfying (5) and the Dirichlet boundary condition we call the constants $\lambda_j$ eigenvalues and the solutions $u_j$ eigenfunctions, respectively.
In [1] we proved an existence theorem for the eigenvalue problem:

**Theorem 1** There exist countably many number of distinct normalized eigenfunctions with associated eigenvalues to the eigenvalue problem (5).

Moreover for the eigenvalues \( \lambda_j \) the next theorem is valid:

**Theorem 2** For the eigenvalues \( \lambda_j \) of the Dirichlet eigenvalue problem of (5) the relation \( \lambda_j \to \infty \) holds when \( j \to \infty \).

Let us denote by \( \lambda_1(D) \) the smallest eigenvalue concerning the domain \( D \) and by \( \lambda_1(B) \) the smallest eigenvalue concerning the ball \( B \) with center at the origin such that \( \text{mes}(B) = \text{mes}(D) \). Then it was proved by F. de Thelin (see [9] p. 356, Theorem 2) that

\[
\lambda_1(D) \geq \lambda_1(B)
\]

holds for the problem (5).

In the recent years many papers were published on the Eq. (5), among them M. Otani [5]; S. Sakaguchi [7]; Jie Jiang [4].

### 3. THE EIGENVALUE PROBLEM FOR A RECTANGLE

We consider the Eq. (5) which one is a quasilinear equation of divergence form in two variables

\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) + \lambda u^p = 0 \quad \text{in} \ D. \tag{6}
\]

Let \( D \in \mathbb{R}^2 \) be a bounded domain with smooth boundary \( \partial D \). The boundary condition corresponding to the Dirichlet problem is

\[
u|_{\partial D} = 0
\]

and the eigenvalues and eigenfunctions are denoted by \( \lambda^{(1)} \) and \( u^{(1)} \), respectively. The boundary condition of the Neumann problem is

\[
\frac{\partial u}{\partial n} \bigg|_{\partial D} = 0,
\]

\( \lambda^{(2)} \) and \( u^{(2)} \) denote the eigenvalues and eigenfunctions, respectively.
EIGENVALUE PROBLEM

It was proved in [1] that the Dirichlet problem of (5) has at least one nonnegative weak solution \( u \in W^{1,p-1}_0(D) \) in \( \mathbb{R}^N \). We shall show that the Dirichlet and Neumann problem of (6) have solutions belonging to \( C^2(\Omega) \cap C^1(D\Omega) \) when \( \Omega \) is bounded by the rectangle

\[
\Omega = \{(x,y): 0 \leq x \leq a, 0 \leq y \leq b\}.
\]

**Theorem 3** For the Dirichlet problem of (6)

\[
\lambda^{(1)}_{k,l} = p\pi^{p+1}\left(\frac{k^{p+1}}{a^{p-1}} + \frac{l^{p+1}}{b^{p+1}}\right),
\]

\[
u^{(1)}_{k,l} = A_{k,l}S_p\left(\frac{k\pi}{a}x\right)S_p\left(\frac{l\pi}{b}y\right), \quad k,l = 1,2,\ldots
\]

are eigenvalues and eigenfunctions, respectively and for the Neumann problem of (6)

\[
\lambda^{(2)}_{k,l} = p\pi^{p+1}\left(\frac{k^{p+1}}{a^{p-1}} + \frac{l^{p+1}}{b^{p+1}}\right),
\]

\[
u^{(2)}_{k,l} = B_{k,l}S_p\left(\frac{k\pi}{a}x + \frac{\pi}{2}\right)S_p\left(\frac{l\pi}{b}y + \frac{\pi}{2}\right), \quad k,l = 0,1,2,\ldots
\]

are eigenvalues and eigenfunctions, respectively. \( \Omega \) is bounded by the rectangle \( \Omega = \{(x,y): 0 \leq x \leq a, 0 \leq y \leq b\} \) and \( \pi = \frac{2\pi}{p+1} \sin\frac{\pi}{p+1} \), \( A_{k,l} = \text{const.} \), \( B_{k,l} = \text{const.} \), and function \( S_p(x) \) is the solution of the differential equation \( S''_pS''_p + S'_p = 0 \) under condition \( S_p(0) = S_p(\pi) = 0 \).

**Conjecture** If the domain \( \Omega \) is rectangular then all the solutions of the Dirichlet and Neumann problem of (6) have the multiplicative form \( u(x,y) = r(x)q(y) \).

**Proof** We look for the solution \( u^{(1)} \) of the Dirichlet eigenvalue problem of (6) in the form

\[
u(x,y) = r(x)q(y).
\]

Substituting it to the Eq. (6) we get

\[
[(r')^p]'q^p + [(q')^p]'r^p = 0,
\]
where $x \in (0, a)$, $y \in (0, b)$. We suppose that $u^{(1)}(x, y) = r(x)q(y) \neq 0$ at some point $(x, y)$. From this equation we have got
\[ p \frac{|r'|^{p-1}r''}{r^p} = -\lambda - p \frac{|q'|^{p-1}q''}{q^p}. \]

For the functions $r(x)$ and $q(y)$ we get the following half-linear second order ordinary differential equations
\begin{align*}
r''|r'|^{p-1} + \frac{\mu}{p} r^\mu = 0 & \quad \text{if } x \in (0, a), \quad (7) \\
q''|q'|^{p-1} + \frac{\nu}{p} q^\nu = 0 & \quad \text{if } y \in (0, b), \quad (8)
\end{align*}

and $\lambda = \mu + \nu$. Now the Dirichlet boundary condition can be written as
\[ r(0) = r(a) = 0 \quad \text{and} \quad q(0) = q(b) = 0. \quad (9) \]

A solution of the problem (7), (9) is given by A. Elbert [3] as $r(x) = S_p(x)$ in the case $\mu = p$, where
\[ x = \int_0^{S_p} \frac{d\sigma}{r(x \sigma^{p+1})} \quad \text{in } [0, \frac{\tilde{\pi}}{2}] \quad \text{and} \quad \tilde{\pi} = \frac{2\pi}{p+1} \sin \frac{\pi}{p+1}. \]

$S_p(x)$ can be extended to the whole real axis:
\[ S_p(x) = \begin{cases} 
S_p(\tilde{\pi} - x) & \text{if } \frac{\tilde{\pi}}{2} \leq x < \tilde{\pi}, \\
-S_p(x - \tilde{\pi}) & \text{if } \tilde{\pi} \leq x < 2\tilde{\pi}, \\
S_p(x + 2k\tilde{\pi}) & \text{where } k = \pm 1, \pm 2, \ldots.
\end{cases} \]

Furthermore we have that
\[ S_p(x) = 0, \quad \text{if } x = 0, \pm \tilde{\pi}, \pm 2\tilde{\pi}, \ldots \]
\[ S'_p(x) = 0, \quad \text{if } x = 0, \pm \frac{\tilde{\pi}}{2}, \pm 3\frac{\tilde{\pi}}{2}, \ldots. \]
The function $S_p(x)$ plays the same role in the case of half-linear differential equations as $\sin x$ in the case of linear differential equations. We have the relation

$$|S_p(x)|^{p+1} + |S'_p|^{p+1} = 1 \quad x \in \mathbb{R}$$

which can be considered as a generalization of the Pythagorean relation.

In the case $\mu \neq p$ by substituting of $r(x) = S_p(cx)$ we get

$$S''|S'_p|^{p-1} + \frac{\mu}{pc^{p+1}} S'_p = 0, \quad S_p(0) = S_p(a) = 0$$

and necessarily

$$\mu = pc^{p+1}.$$ 

Satisfying the boundary condition we find

$$c = \frac{k\pi}{a} \quad (k \in \mathbb{N}) \quad \text{and} \quad \mu = p \frac{k^{p+1}\pi^{p+1}}{a^{p+1}}.$$

Apart from a constant factor the function $r(x)$ can be only one of the functions

$$r_k(x) = S_p\left(\frac{k\pi}{a}x\right).$$

The solution of the problem (8), (10) could be found analogously

$$q_l(y) = S_p\left(\frac{l\pi}{b}y\right) \quad (l \in \mathbb{N}) \quad \text{and} \quad \nu = p \frac{l^{p+1}\pi^{p+1}}{b^{p+1}}.$$

The eigenvalue $\lambda^{(1)}_{k,l}$ is obtained by $\lambda = \mu + \nu$.

The solution of the Neumann problem can be determined in the same way. Thus we have the eigenfunctions $u^{(2)}_{k,l}(x,y)$ and the corresponding eigenvalues $\lambda^{(2)}_{k,l}$. We remark that also the values $k = 0$, $l = 0$ are admitted in the expressions of $\lambda^{(2)}_{k,l}$ and $u^{(2)}_{k,l}(x,y)$ while in the case of Dirichlet problem no.

All the solutions of the Dirichlet and Neumann problem, which are of product form has been found. We known in the linear case $p = 1$
that the functions \( \{u_{k,l}\}_{k=1,l=1}^{\infty,\infty} \) form a complete system in \( L^2(0, \pi) \). The same is not known for the non-linear case but we hope so.

4. RADIAL SYMMETRIC SOLUTIONS

Let us denote by \( \lambda_1(D) \) the smallest eigenvalue concerning the ball \( B \) with center at the origin such that \( \text{mes}(B) = \text{mes}(D) \). Then it is known that

\[
\lambda_1(D) \geq \lambda_1(B)
\]

holds for the problem (1)-(2). Concerning the ball the usual method is the introduction of polar coordinates.

For the problem (5) we shall define the distance \( \rho(\geq 0) \) between the point and the origin in \( \mathbb{R}^N \) as follows:

\[
\rho^{\frac{1}{p+1}} = \sum_{i=1}^{N} |x_i|^{\frac{1}{p+1}}.
\]

In the case \( \rho = 1 \) the Eq. (12) gives the equation of the unit "ball" \( B_\rho \) in \( \mathbb{R}^N \). We mention that the curve \( \rho = 1 \) in \( \mathbb{R}^2 \) is a central symmetric convex curve (see Fig. 1) which plays the same role in the case of nonlinear differential equation (5) as the unit circle in the case of linear \( (p = 1) \) partial differential equation.

For the "ball" \( B_\rho \) we introduce now instead of rectangular coordinates \( F_D \) a new type of polar coordinates \( \rho, \varphi_1, \ldots, \varphi_{N-1} \) as

![Figure 1](image_url)
follows
\[ x_1 = \rho \prod_{i=1}^{N-1} [S'(\varphi_i)], \]
\[ x_k = \rho [S(\varphi_{k-1})]^{N-1} \prod_{i=1}^{N-1} [S'(\varphi_i)], \quad \text{if } 1 < k \leq N \]
where \( S = S(\varphi_i), 1 \leq i \leq N-1 \) is the generalized sine function. Here \( S' = dS(\varphi)/d\varphi \).

The transformation has the Jacobian
\[ \Delta = \frac{\partial(\rho, \varphi_1, \ldots, \varphi_{N-1})}{\partial(x_1, x_2, \ldots, x_N)}. \]
where we have the following relations for the partial derivatives of \( \rho_{x_k} \) with respect to \( x_k \) (\( 1 \leq k \leq N \)):
\[ \sum_{i=1}^{N} |\rho_k|^{p+1} = 1, \]
and
\[ \rho_1 = \prod_{i=1}^{N-1} [S'(\varphi_i)]^{1/p}, \]
\[ \rho_k = [S(\varphi_{k-1})]^{1/p} \prod_{i=k}^{N-1} [S'(\varphi_i)]^{1/p}, \quad \text{if } 1 < k \leq N \]
and for the partial derivatives \( \varphi_{ik} \) of \( \varphi_i \) with respect to \( x_k \) (\( 1 \leq i \leq N-1, 1 \leq k \leq N \)) by using notation
\[ \varphi_{11} = -\frac{1}{\rho} \frac{S'(\varphi_1)}{|S'(\varphi_1)|^{1/p-1}} \prod_{j=2}^{N-1} \frac{1}{S'(\varphi_j)}, \]
\[ \varphi_{ik} = \begin{cases} 0, & \text{if } k > i + 1 \\ \frac{1}{\rho} \frac{S'(\varphi_i)}{|S'(\varphi_{i-1})|^{1/p-1}} \prod_{j=i+2}^{N-1} \frac{1}{S'(\varphi_j)}, & \text{if } k = i + 1 \\ -\frac{1}{\rho} \frac{S(\varphi_{k-1})}{|S'(\varphi_{k-1})|^{1/p-1}} \left( \prod_{j=k}^{i-1} [S'(\varphi_j)]^{1/p} \right) \frac{S(\varphi_k)}{|S'(\varphi_k)|^{1/p-1}} \prod_{l=i+2}^{N-1} \frac{1}{S'(\varphi_l)}, & \text{if } k < i + 1 \end{cases} \]
For the Dirichlet problem of (5) in $\mathbb{R}^2$ the inequality (11) also holds. It can be proved by the rearrangement method [1] where the "ball" $B_p$ is defined by

$$B_p = \left\{ (x_1, \ldots, x_N) : \left[ \sum_{i=1}^{N} |x_i|^{p+1} \right]^{p/(p+1)} \leq 1 \right\}, \quad 0 < p < \infty.$$ 

This proof can be generalized to $\mathbb{R}^N$.

**Theorem 4** Let the unit "ball" $B_p$ in $\mathbb{R}^N$ be defined by

$$B_p = \left\{ (x_1, \ldots, x_N) : \left[ \sum_{i=1}^{N} |x_i|^{p+1} \right]^{p/(p+1)} \leq 1 \right\}, \quad 0 < p < \infty$$

and let $u \in W^{1,p+1}_0(B_p)$ be a nonnegative eigenfunction associated with the first eigenvalue $\lambda_1$ of the eigenvalue problem (5) in $B_p$ under boundary condition $u|_{\partial B_p} = 0$. Then for the radially symmetric function $u(x) = v(\rho)$ the eigenvalue problem becomes

$$\frac{\partial}{\partial \rho} \left( \frac{\partial v}{\partial \rho} \right)^{\rho} + \frac{N-1}{\rho} \left( \frac{\partial v}{\partial \rho} \right)^{\rho} + \lambda v^{\rho} = 0,$$

$$v(1) = 0.$$ 

For the proof see [2].

**Remark 5** The radially symmetric solutions $u(x) = v(\rho)$ of (4) in the domain

$$B = \left\{ (x_1, \ldots, x_N) : \left[ \sum_{i=1}^{N} x_i^2 \right]^{1/2} \leq 1 \right\},$$

and the radially symmetric solutions of (5) in the domain

$$B_p = \left\{ (x_1, \ldots, x_N) : \left[ \sum_{i=1}^{N} |x_i|^{p+1} \right]^{p/(p+1)} \leq 1 \right\}, \quad 0 < p < \infty$$

satisfy formally the same equation, namely the Eq. (13).
W. Reichel and W. Walter [6] have proved that the initial value problem of (13) is uniquely solvable.

4.1. The Determination of the First Eigenvalues

We shall consider the eigenvalue problem (13) with the Dirichlet condition. It means that the boundary condition (2) joins to the problem (4) or (5). In the radially symmetric case the condition (2) is equivalent to the conditions:

\[ v(1) = 0, \quad v'(0) = 0. \] (14)

Instead of these condition we shall look for the solutions of (13) under conditions: \( v(0) = 1, \quad v'(0) = 0. \)

By the transformation

\[ v(\rho) = x(t), \quad \text{where } t = \left( \frac{\sqrt[\rho]{\lambda \rho}}{(p+1-N)/p} \right) \]

we obtain the system of differential equations

\[ x' = y^{1/p}, \]

\[ y' = -\left( \frac{p}{p+1-N} \right)^{p+1} t^{(p+1)(N-1)/(p+1-N)} x^{p}, \]

under initial conditions

\[ x(0) = 1, \]
\[ y(0) = 1. \]

When \( p > N - 1 \) this initial value problem can be solved by the fourth order Runge-Kutta method. The first zero of the function \( x(t) \) denoted by \( t_1 \) will be determined. The condition (14) can be satisfied by the substitution of

\[ r^{\sqrt{\lambda} \rho} = t_1^{p/(p+1-N)} \]

at \( \rho = 1 \). In such a way the values of \( r^{\sqrt{\lambda}} \) will be determined for several values of \( N \) as a function of \( p \) (see Fig. 2).
4.2. An Upper Approximation of the Smallest Eigenvalues

We note that the Runge-Kutta method does not work if $0 < p \leq N - 1$. So, in this case we have to find another method for the determination of the first eigenvalue. From [1] it is known that the smallest, first eigenvalue can be determined as the solution of the variational problem:

$$\lambda_1 = \inf R[v],$$

where $R[v]$ is called the generalized Rayleigh quotient and for the differential equation (13) it has the form

$$R[v] = \frac{\int_a^b [\rho |v|^p + (N - 2)vv'_{\rho}] d\rho}{\int_a^b \rho |v|^{p+1} d\rho}, \quad \rho \in [a, b]. \quad (15)$$

Our goal is to determine the infimum of $R[v]$ and a corresponding function among the admissible functions $v = v(\rho)$ vanishing on $\partial D$. Choosing a special set of test functions

$$v_\alpha = 1 - \rho^{(\frac{1}{\alpha} - 1)} \alpha > 0,$$

$$v_\alpha(0) = 1,$$

$$v'_\alpha(0) = 0.$$
Substituting the test function \( v_\alpha \) into the expression of (15) we get for the Rayleigh quotient

\[
R[v_\alpha] = \left(\frac{p + 1}{p}\right)^{p} \int_0^1 \rho \left(\frac{(p+1)/p}{\alpha - 1}(p+1)+N-1\right) \rho^{N-1}(1 - \rho^{a(p+1)/p})^{p+1} d\rho.
\]

Evaluating the integrals in \( R[v_\alpha] \) we obtain the following expression

\[
R[v_\alpha] = \frac{(p+1/\alpha)^{p+2}}{\left[\left(\frac{p+1}{\alpha - 1}\right)(p+1)+N\right]B\left(\frac{Np}{(p+1)a}, p+2\right)}.
\]

where \( B(x, y) \) is the Beta function \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \). We want to find numerically the minimum value of \( R[v_\alpha] \). Here we use the approximation for the calculation of the Gamma function \( \Gamma(x) \) as follows:

\[
\Gamma(x) = \sqrt{\frac{2\pi}{x}} e^{x \ln x - f(x)}
\]

\[
\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)} \quad \text{if} \ x + n \geq 5,
\]

\[
f(x) = \frac{1}{1 + 1/(3x(4x+1/(5(1/(6x) - x))))} \quad \text{if} \ x \geq 5.
\]

For fixed value of the dimension \( N \), and \( p \ (0 < p < \infty) \) we can evaluate the minimum value of \( R[v_\alpha] \) and also the approximation value of \( r\sqrt{\lambda_1} \). In Figure 3, the values of \( r\sqrt{\lambda_1} \) are plotted as the function of \( p \) in two dimensions.

\[FIGURE 3 \ Values of r\sqrt{\lambda_1}.\]
The solution of the Dirichlet and Neumann problem for the nonlinear equation (13), to find eigenfunctions and corresponding eigenvalues, can be solved also by the Runge-Kutta method. The large eigenvalues can be calculated numerically from the nonlinear system of differential equations.

References


