ON BERNSTEIN-NIKOL'SKII INEQUALITIES AND WIDTHS OF SOBOLEV FUNCTION CLASSES

UDC 517.5

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Let $I$ be the unit interval $[0,1]$, $1 \leq p, q \leq \infty$, $r \in \mathbb{N}$, $x: I \to \mathbb{R}$, $\Gamma_i, i = 0 \vee 1$, the boundary conditions

$$(x|_{\partial I} \in \Gamma_i) \Leftrightarrow (x(i) = \cdots = x^{(r-1)}(i) = 0)$$

for functions possessing the corresponding derivatives,

$$W_p^r(\Gamma_i) = \{ x: I \to \mathbb{R}, x^{(r-1)}(\cdot) \text{ abs. cont.}, \| x^{(r)} \|_{L_p(I)} \leq 1, \ x|_{\partial I} \in \Gamma_i \}.$$ 

In [1] and [2] Kolmogorov $n$-widths of the class $W_p^r(\Gamma)$ (for $\Gamma = \Gamma_i, i = 0 \vee 1$, and other boundary conditions) are examined in the $L_q$ metric for $p \geq q$. These quantities $d_n(W_p^r(\Gamma), L_q), p \geq q$, turned out to be closely related to the spectrum of the differential equation

$$(-1)^r((x^{(r)}(p))^{(r)} - \lambda^q x(p)) = 0,$$

where $1 < p, q < \infty$, $(x(p)) := |x|^{q-1} \text{ sgn} x$, and the $\Gamma_i^q$ are the conjugate boundary conditions ($\Gamma_0^q = \Gamma_0$, $\Gamma_1^q = \Gamma_1$).

We denote by $\text{SP}_n(r, p, q, \Gamma_i), n \in \mathbb{Z}_+$, the collection of those spectral pairs $(\lambda, x)$ of the problem (1) for which the number of zeros $N(x) = n$, and we denote by $(\hat{\lambda}_n, \hat{x}_n)$ and $(\tilde{\lambda}_n, \tilde{x}_n)$ respectively the maximal and minimal spectral pairs in $\text{SP}_n(r, p, q, \Gamma_i)$, i.e.,

$$(\hat{\lambda}_n, \hat{x}_n) \in \text{SP}_n(r, p, q, \Gamma_i), \quad \hat{\lambda}_n = \max \{ \lambda: (\lambda, x) \in \text{SP}_n \},$$

$$(\tilde{\lambda}_n, \tilde{x}_n) \in \text{SP}_n(r, p, q, \Gamma_i), \quad \tilde{\lambda}_n = \min \{ \lambda: (\lambda, x) \in \text{SP}_n \}.$$

The correctness of these definitions follows from [2]. Moreover, it was established there that either $\hat{\lambda}_n = \tilde{\lambda}_n$ (for example, this obviously is the case when $1 < p = q < \infty$ [1]), and then $\text{SP}_n(r, p, q, \Gamma_i)$ consists of one element, or else $\hat{\lambda}_n = \tilde{\lambda}_n, 1 < p, q < \infty$, and then to every eigenvalue $\lambda_n \in \text{SP}_n$ corresponds an eigenfunction unique up to normalization (unit multiplicity of the spectrum).

The object of this note is the proof of the following theorem about Bernstein widths.

Theorem. Suppose $r \in \mathbb{N}, n \in \mathbb{Z}_+, 1 < p < q < \infty$, and $\Gamma = \Gamma_i, i = 0 \vee 1$. Then

$$b_n(W_p^r(\Gamma_i), L_q) = \lambda_n^{-1}.$$
2. Estimation of the width $b_n$ from below \((b_n(W^{r}(\Gamma), L_{q}) \geq \lambda_{n}^{-1})\)

We use the construction of Pinkus [1], who proved an analogous statement when
\(p = q\). It follows from [2] that the zeros \(\tilde{x}_{n}\) are simple, and \(\tilde{x}_{n}^{(p)}\) has \(n\) changes of
sign on \(I\) at points \(\mathcal{F} = \{0 < \tau_{1} < \cdots < \tau_{n} < 1 = \tau_{n+1}\}\), and that \(\|\tilde{x}_{n}^{(p)}\|_{L_{p}(I)} = 1\)
and \(\lambda_{n}^{-1} = \|\tilde{x}_{n}\|_{L_{q}(I)}\). Put

\[ f_{i}(t) = \{\tilde{x}_{n}^{(p)}(t), \; \tau_{i-1} < t \leq \tau_{i}, \; 0, \; t \notin (\tau_{i-1}, \tau_{i}]\}, \quad 1 \leq i \leq n + 1. \]

We shall examine the extremal problem

\[
\min_{c \in \mathbb{R}^{n+1}} \left( \frac{\left\| \sum_{i=1}^{n+1} c_{i} F_{i}(t) \right\|_{L_{q}}}{\left\| \sum_{i=1}^{n+1} c_{i} f_{i}(t) \right\|_{L_{p}}} \right),
\]

where \(F_{i}|_{\Omega} \in \Gamma_{0}\) and \(F_{i}^{(p)}(t) = f_{i}(t)\). The following properties of the problem (3)
transfer without changes from [1] \((p = q)\):

(a) There exists an extremal vector \(c^{*}\) in (3), and moreover

\[
\left\| \sum_{i=1}^{n+1} c^{*}_{i} F_{i}(t) \right\|_{L_{q}} \left/ \left\| \sum_{i=1}^{n+1} c^{*}_{i} f_{i}(t) \right\|_{L_{p}} \right. = : \mu_{n} > 0.
\]

(b) The coordinates of the vector \(c^{*}\) are nonzero and alternate in sign.

(c) The necessary conditions for an extremum lead to the system of equations

\[
\int_{0}^{1} F_{k}(t)((\mathcal{F}(t))_{q}) d t = \mu_{n}^{(p)}(c^{*}_{k})_{(p)} \|f_{k}\|_{L_{p}(I)},
\]

where \(\mathcal{F}^{*}(t) = \sum_{i=1}^{n+1} c_{i} F_{i}(t)\) and \(\sum_{i=1}^{n+1} |c^{*}_{i}|^{p} \|f_{i}\|_{L_{p}}^{p} = 1\), \(c^{*}_{i} > 0\).

(d) The vector \(\bar{c} = (1, -1, \ldots)\) and the function \(\bar{F}(t) = \sum_{i=1}^{n+1} \bar{c}_{i} F_{i}(t) = \tilde{x}_{n}(t)\)
satisfy the system

\[
\int_{0}^{1} F_{k}(t)(\bar{F}(t))_{(q)} d t = \lambda_{n}^{-q}(\bar{c}_{k})_{(p)} \|f_{k}\|_{L_{p}(I)}^{p}.
\]

The inequality \(\mu_{n} \leq \lambda_{n}^{-1}\) is obvious. We shall assume that

\[
\mu_{n} < \lambda_{n}^{-1}.
\]

We multiply (4) by \(e^{q-1}, \; e > 0\), and subtract from (5) the left and right sides
respectively:

\[
\int_{0}^{1} F_{k}(t)((\mathcal{F}(t))_{(q)} - (e \mathcal{F}(t))_{(q)}) d t = \|f_{k}\|_{L_{p}(I)}^{p} (\lambda_{n}^{-q}(\bar{c}_{k})_{(p)} - \mu_{n}^{(p-q)} e^{q-p}(c^{*}_{k})_{(p)}).
\]

We put \(e^{*} = \min_{1 \leq i \leq n+1} (\bar{c}_{i} / c^{*}_{i})\) in (7). By virtue of (b) and (c) we have \(0 < e^{*} < 1\),
and hence the number of sign changes

\[
P(\lambda_{n}^{-q}(\bar{c}_{k})_{(p)} - \mu_{n}^{(p-q)} e^{q-p}(c^{*}_{k})_{(p)})
= P(\bar{c}_{k} - (\mu_{n}/\lambda_{n}^{-1})^{q/(p-1)}(e^{*})^{(q-p)/(p-1)}c^{*}_{k}) = P(\bar{c}_{k}) = n,
\]
or \((\mu_{n}/\lambda_{n}^{-1})^{q/(p-1)} < 1\) and \((e^{*})^{(q-p)/(p-1)} < e^{*} < 1, \; q > p\).
On the other hand, by virtue of Rolle's theorem
\[ P(\tilde{F}(\cdot) - \varepsilon^* F^*(\cdot)) \leq N(\tilde{F}(\cdot) - \varepsilon^* F^*(\cdot)) \leq P(\tilde{F}(\cdot) - \varepsilon^* F^*(\cdot)) \]
\[ \leq P(\tilde{c} - \varepsilon^* c^*) \leq n - 1, \]
which contradicts (8). Hence, the assumption (6) was false, and \( \mu_n = \frac{1}{\lambda_n} \).

3. Estimation of the width \( b_n \) from above \((b_n(W^*_p(\Gamma_1), L_q) \leq \frac{1}{\lambda_n} )\)

Let us suppose that there exists a linearly independent system \( \{ f_1, \ldots, f_{n+1} \} \) of measurable functions on \( I \) for which
\[
\min_{c \in \mathbb{R}^{n+1}} \left( \left\| \sum_{i=1}^{n+1} c_i \overline{f}_i(t) \right\|_{L_q} / \left\| \sum_{i=1}^{n+1} c_i \bar{f}_i(t) \right\|_{L_p} > \frac{1}{\lambda_n} \right),
\]
where \( \overline{f}_i = \bar{f}_i \) and \( F_i|_{\Gamma_1} \in \Gamma_0 \), i.e.,
\[
\left\| \sum_{i=1}^{n+1} c_i \overline{f}_i(t) \right\|_{L_q} > \frac{1}{\lambda_n}
\]
for any function from the \( n \)-dimensional sphere
\[
S_n = \left\{ F(t) = \sum_{i=1}^{n+1} c_i \overline{f}_i(t) \, | \, \left\| \sum_{i=1}^{n+1} c_i \bar{f}_i(t) \right\|_{L_p} = 1 \right\}.
\]

Let \( x_0(\cdot) \in S_n \). We shall construct a sequence of function \( x_k(t), \, k \in \mathbb{N} \), according to the rule
\[
(y_k)^{(r)}(t) = (-1)^{r+1} \hat{y}_k(x_{k-1}(t)), \quad y_k|_{\Gamma_1} \in \Gamma_1^T, \quad x_k^{(r)}(t) = (y_k)^{(r)}(t), \quad x_k|_{\Gamma_1} \in \Gamma_1,
\]
where \( \hat{y}_k > 0 \) are determined from the normalization condition \( \|x_k^{(r)}\|_{L_q} = 1 \).

We define \( S_n(k) = \{ x_k(\cdot, x_0), \, x_0(\cdot) \in S_n \} \). We shall use the following properties of the transformation (11), which follow from [2].
(e) \( \|x_k\|_{L_q} \) is a nondecreasing function of the argument \( k \in \mathbb{N} \), and for each \( k \in \mathbb{N} \) there is an \( x_k \in S_n(k) \) with \( n \) zeros inside \( I \).
(f) \( \lim_{k \to \infty} S_n(k) \) consists of eigenfunctions (1), of which at least one has no fewer than \( n \) zeros.
(g) \( \lambda_n \) is monotonically increasing for \( n \in \mathbb{N} \).

By \( \bar{c} \in \mathbb{R}^{n+1} \) we denote a vector such that the function \( \bar{x}_0(t) = \sum_{i=1}^{n+1} \bar{c}_i \bar{f}_i(t) \), under the transformation (11) and as \( k \to \infty \), goes into an eigenfunction with \( N \geq n \) zeros. We have
\[
\min_{c \in \mathbb{R}^{n+1}} \left( \left\| \sum_{i=1}^{n+1} c_i \overline{f}_i(t) \right\|_{L_q} / \left\| \sum_{i=1}^{n+1} c_i \bar{f}_i(t) \right\|_{L_p} \right) \leq \|\bar{x}_0(t)\|_{L_q}
\]
\[
\leq \lim_{k \to \infty} \|x_k(t, \bar{x}_0(\cdot))\|_{L_q} = \lambda_{N-1} \leq (\lambda_N)^{-1} \leq (\lambda_n)^{-1},
\]
which contradicts (10). This means that the assumption (10) cannot be satisfied and that the estimation of the Bernstein width from above is correct. The theorem is proved.
The extremal problems of the form (3) are equivalent to the search for the best constant in the corresponding Bernstein-Nikol'skii inequality. The history of the problem of widths is expounded in detail in [3] and [2] and the connection with stationary points of the Rayleigh relation has its beginning in Tikhomirov's papers. For \( p = q \) the theorem was proved by Pinkus [1], where also other widths are investigated. These are discrete versions of our theorem (see [4] and [5]). It is clear that our scheme of proof can be applied in broad outline in the case of other boundary conditions \( \Gamma [2] \); however, we considered it preferable not to burden our paper with that.

Let us note also that on the set of parameters \( CQ = \{0 \leq p^{-1} \leq 1, 0 \leq q^{-1} \leq 1\} \), with \( n \) fixed, it is possible to extend the relations

\[
\begin{align*}
d_n(W_p^r, L_q) &= \hat{\lambda}_n^{-1}, \quad p \geq q, \\
b_n(W_p^r, L_q) &= \hat{\lambda}_n^{-1}, \quad p \leq q
\end{align*}
\]

(respectively the top and bottom triangles of \( CQ \)) over the demarcation line of the diagonal \( p = q \). The quantitative estimation of such penetration, and the qualitative analysis of the catastrophes at the moment when the relations (12) collapse, ought to be the subject of a separate paper.

BIBLIOGRAPHY