ON THE APPROXIMATION NUMBERS OF SOBOLEV EMBEDDINGS ON SINGULAR DOMAINS AND TREES

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1. Introduction

Let Ω be a domain (i.e. a connected open set) in \(\mathbb{R}^n\), \(n \geq 1\) of finite volume |Ω|, and let \(E_C(\Omega) : W^{1,p}_C(\Omega) \hookrightarrow L^p_C(\Omega)\), \(E_0 : W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)\) be the natural embeddings, where \(W^{1,p}_C(\Omega)\), \(L^p_C(\Omega)\) are the quotient spaces \(W^{1,p}(\Omega)/C\), \(L^p(\Omega)/C\) respectively, and \(W^{1,p}(\Omega), W^{1,p}_0(\Omega)\) are the standard Sobolev spaces, with \(1 < p < \infty\). The approximation numbers \(a_m(E_C(\Omega)), m \in \mathbb{N}\), of \(E_C(\Omega)\) are defined as

\[
a_m(E_C(\Omega)) := \inf\{\|E_C(\Omega) - P\|_{W^{1,p}_C(\Omega) \to L^p_C(\Omega)} : \text{rank} P < m\}
\]

and, for \(\varepsilon > 0\) we set

\[
\nu_C(\varepsilon, \Omega) := \max\{m : a_m(E_C(\Omega)) \geq \varepsilon\}.
\]

The objective of this paper is to determine, for a wide class of domains \(\Omega\) (including ones with highly irregular, even fractal, boundaries), upper and lower bounds for \(\nu_C(\varepsilon, \Omega)\) which can readily be estimated in examples. Roughly speaking, the core of our \(\Omega\) is the union of rectangular blocks or other regular polyhedra, and the remainder is made up of sets of small diameter and irregular sets which are generalised ridged domains (GRDs); see Definition 1.1 below. The latter portions include regions near parts of the boundary of \(\Omega\) which may be horns, cusps, spirals or fractal curves. A domain \(\Omega\) which does not fall within this category would be highly pathological! The lower bound for \(\nu_C(\varepsilon, \Omega)\) is determined in terms of the number

\[
\mu_0(\varepsilon, \Omega) := \max\{\dim S : \alpha(S) \leq 1/\varepsilon\}
\]

where \(S\) is a subset of \(W^{1,p}_0(\Omega)\) of finite dimension \(\dim S\) and

\[
\alpha(S) := \sup_{f \in S} \frac{\|\nabla f\|_{L^p(\Omega)}}{\|f\|_{L^p(\Omega)}}.
\]

When \(p = 2\), \(\nu_C(\varepsilon, \Omega) = N(\varepsilon^{-2}, -\Delta_{N,\Omega})\), the counting function for the positive eigenvalues of the Neumann Laplacian \(-\Delta_{N,\Omega}\), and \(\mu_0(\varepsilon, \Omega) = N(\varepsilon^{-2}, -\Delta_{D,\Omega})\), where \(-\Delta_{D,\Omega}\) is the Dirichlet Laplacian on \(\Omega\); see [7, Lemma 5.2; note that in our present notation, \(\nu_{\Omega}(\varepsilon), \mu_0(\varepsilon, \Omega)\) in [7] are denoted by \(\nu_C(\varepsilon, \Omega), \mu_0(\varepsilon, \Omega)\) respectively].
In [7, Section 5], an analogue of Dirichlet-Neumann bracketing is developed for \( \nu\mathcal{C}(\epsilon, \Omega), \mu_0(\epsilon, \Omega) \) and related quantities, for \( 1 < p < \infty \). This forms the basis of the method used in this paper. Another significant feature of the analysis is the role of Hardy-type operators on trees associated with those portions of \( \Omega \) which are GRDs. This allows recent work on the approximation numbers of such operators to be applied.

We refer to [5,7] for a detailed description of generalised ridged domains (GRDs) and their properties, and also to a discussion of the analysis on trees. Here we just give a brief summary. Let \( \Gamma \) be a tree, i.e. a connected graph without loops or cycles, where the edges are non-degenerate closed line segments whose end-points are the vertices. The distance, \( \text{dist}(x,y) \), between \( x,y \in \Gamma \) is defined to be the length of the unique polygonal path which joins \( x \) and \( y \), and this defines a metric topology on \( \Gamma \). When endowed with the natural 1-dimensional Lebesgue measure, \( \Gamma \) is a \( \sigma \)-finite measure space. The path joining two points \( x,y \in \Gamma \) may be parametrized by \( s(t) := \text{dist}(x,t) \), and for \( g \in L^1_{\text{loc}}(\Gamma) \) we have
\[
\int_x^y g(t) \, dt := \int_0^{\text{dist}(x,y)} g(t(s)) \, ds;
\]
see [7, section 2] for further details.

**Definition 1.1** A domain \( \Omega \) is called a *generalized ridged domain* (GRD) if there exist real-valued functions \( u, \rho, \tau \), a tree \( \Gamma \) and positive constants \( \alpha, \beta, \gamma, \zeta \) such that the following conditions are satisfied:

- \( u : \Gamma \to \Omega, \rho : \Gamma \to \mathbb{R}^+ \equiv (0, \infty) \) are Lipschitz;
- \( \tau : \Omega \to \Gamma \) is surjective and for each \( x \in \Omega \), there exists a neighbourhood \( V(x) \) such that for all \( y \in V(x), |\tau(x) - \tau(y)|_{\Gamma} \leq \gamma|x-y| \), where \( |\cdot|_{\Gamma} \) denotes the metric on \( \Gamma \); thus \( \tau \) is uniformly locally Lipschitz;
- \( |x - u \circ \tau(x)| \leq \alpha \rho \circ \tau(x) \) for all \( x \in \Omega \);
- \( |u'(t)| + |\rho'(t)| \leq \beta \) for all \( t \in \Gamma \);
- with \( B_t := B(u(t), \rho(t)) \) and \( \mathcal{C}(x) := \{ y : sy + (1-s)x \in \Omega \text{ for all } s \in [0,1] \} \), we have that for all \( x \in \Omega, B_{\tau(x)} \subset \Omega \) and \( \mathcal{C}(x) \cap B_{\tau(x)} \) contains a ball \( B(x) \) such that \( |B(x)|/|B_{\tau(x)}| \geq \zeta > 0 \). The curve \( t \mapsto u(t) : \Gamma \to \Omega \) is called the *generalized ridge* of \( \Omega \).

The map \( \tau \) in Definition 1.1 defines a positive Borel measure \( \mu \) on \( \Gamma \) such that for any open subset \( \Gamma_0 \) of \( \Gamma \), \( \mu(\Gamma_0) = |\tau^{-1}(\Gamma_0)| \), and
\[
\int_{\Gamma} F(t) \, d\mu(t) = \int_{\Omega} (F \circ \tau)(x) \, dx;
\]
this is initially defined on compactly supported functions on \( \Gamma \) but extends by continuity to give an isometry \( F \mapsto F \circ \tau \) from \( L^p(\Gamma,d\mu) \).
to $L^p(\Omega)$ when $L^p(\Gamma, d\mu)$ is endowed with the norm

$$
\|F\|_{L^p(\Gamma, d\mu)} := \left\{ \int_{\Gamma} |F(t)|^p d\mu \right\}^{1/p}.
$$

When $\mu$ is replaced by the Lebesgue measure on $\Gamma$ we use the notation $L^p(\Gamma)$.

The paper is organised as follows. Section 2 contains an analysis of the one-dimensional problem when $\Omega$ is an open interval. In this case the norms of $E_C(\Omega)$ and $E_0(\Omega)$ are equal and are determined precisely (Theorem 2.1), and bounds for $\nu_C(\epsilon, \Omega)$ and $\mu_0(\epsilon, \Omega)$ given which yield the same asymptotic limit (Theorem 2.3). Similar results have also been independently obtained recently in [4]. In order to apply results from [7], relationships between $E_C(\Omega)$ and embeddings $E_g(\Omega)$ between subspaces $W_{1,p}^g$ and $L_p^g(\Omega)$ of $W_{1,p}(\Omega)$ and $L_p(\Omega)$ respectively (related to the spaces $W_{-1/2}^g(\Omega)$, $L_p^g(\Omega)$ of [7]) are needed, and these are presented in Section 3. This requires a comparison of various norms on quotient spaces; we note in particular Theorem 3.2 which asserts that the quotient norm on $L_p^C(\Omega)$ is equal to

$$
\min \left\{ \|f - \frac{\int_{\Omega} f g dx}{\int_{\Omega} g dx} \|_{L^p(\Omega)} : g \in L^{p'}(\Omega), \int_{\Omega} g dx \neq 0 \right\}.
$$

Similar results are obtained for spaces defined on trees. The main theorems in which the upper and lower bounds for $\nu_C(\epsilon, \Omega)$ are established, are given in Sections 4 and 5. An example is discussed in Section 6.

The norms on $L^p(\Gamma)$, $L^p(\Omega)$, $W_0^{1,p}(\Omega)$, $W_C^{1,p}(\Omega)$ and $L_C^p(\Omega)$ are denoted respectively by

$$
\|f\|_{p, \Gamma} = \left( \int_{\Gamma} |f(t)|^p dt \right)^{1/p},
$$

$$
\|f\|_{p, \Omega} = \left( \int_{\Omega} |f(x)|^p dx \right)^{1/p},
$$

$$
\|f\|_{1,p, \Omega} = \left( \int_{\Omega} |\nabla f(x)|^p dx \right)^{1/p},
$$

$$
\|[f]\|_{1,p, \Omega} = \left( \int_{\Omega} |\nabla f(x)|^p dx \right)^{1/p},
$$

$$
\|[f]\|_{p, \Omega} = \inf \{\|f - c\|_{p, \Omega} : c \in \mathbb{C} \}.
$$

Note that elements of the quotient spaces $W_C^{1,p}(\Omega)$, $L_C^p(\Omega)$ are equivalence classes, $[\cdot]$, members of the same equivalence class differing by a constant. If the Poincaré inequality is valid on $\Omega$, i.e.

$$
\|f - \frac{1}{|\Omega|} \int f(x) dx\|_{p, \Omega} \leq c \|\nabla f\|_{p, \Omega} \quad (1.4)
$$
for some constant $c$, then the norm of $W^{1,p}_C$ is equivalent to the $W^{1,p}(\Omega)$ norm, namely,

$$\|f\|_{W^{1,p}(\Omega)} := \left\{ \|f\|_{p,\Omega}^p + \|\nabla f\|_{p,\Omega}^p \right\}^{1/p}$$

and $W^{1,p}_C(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$. It is proved in [5, Theorem 2.6] that (1.4) is equivalent to

$$\inf\{\|E(\Omega) - P|W^{1,p}(\Omega) \rightarrow L^p(\Omega)\| : \text{rank}P < \infty\} < 1.$$ 

The norm of a bounded operator $T : X \rightarrow Y$ will be denoted by $\|T\|_{X \rightarrow Y}$, but simplified to $\|T\|$ when the spaces involved are clear.

2. The case $\Omega = (a,b)$

Let $c \in \Omega = (a,b)$, and let $G_c$ be the Hardy operator

$$(G_c f)(x) := \int_c^x f(t) \, dt, \quad f \in L^p(\Omega). \quad (2.1)$$

Then $G_c : L^p(\Omega) \rightarrow L^p(\Omega)$ is bounded with norm $\|G_c|L^p(\Omega) \rightarrow L^p(\Omega)\| \leq |\Omega|$ since

$$\int_\Omega \left( \int_c^x f(t) \, dt \right)^p \, dx \leq \|f\|_{p,\Omega}^p \int_\Omega (\max_{x \in \Omega} |x - c|)^{p-1} \, dx \leq |\Omega|^p \|f\|_{p,\Omega}^p. \quad (2.2)$$

We also have by scaling that

$$\|G_a|L^p(\Omega) \rightarrow L^p(\Omega)\| = \|G_b|L^p(\Omega) \rightarrow L^p(\Omega)\| = |\Omega| \|G_0|L^p(0,1) \rightarrow L^p(0,1)\| = |\Omega|\alpha_p \quad (2.3)$$

say.

**Theorem 2.1** If $c = \frac{(a+b)}{2}$,

$$\|E_0(\Omega)\| = \|E_C(\Omega)\| = \|G_c|L^p(\Omega) \rightarrow L^p(\Omega)\| = \frac{\alpha_p}{2} |\Omega|. \quad (2.4)$$

**Proof.** Given $\varepsilon > 0$, there exists $\phi \in L^p(c, b)$ such that

$$\|G_c\phi\|_{p,(c,b)} \geq \left( \|G_c|L^p(c, b) \rightarrow L^p(c, b)\| - \varepsilon \right)\|\phi\|_{p,(c,b)}$$

$$= \left( \frac{\alpha_p}{2} |\Omega| - \varepsilon \right)\|\phi\|_{p,(c,b)} \quad (2.5)$$

by (2.3). Define the function

$$\psi(t) = \begin{cases} 
\phi(2c - t), & a < t \leq c \\
\phi(t), & c < t < b 
\end{cases}$$
which is symmetrical about the mid-point $c$. Then $\Psi(x) = (G_c \psi)(x)$ is anti-symmetrical about $c$ and we have

\[
\|\psi\|_{p,\Omega} = 2^{1/p} \|\phi\|_{p,(a,b)}
\]
\[
\|\Psi\|_{p,\Omega} = 2^{1/p} \|G_c \phi\|_{p,(a,b)}
\]

These equations and (2.5) yield

\[
\|G_c \psi\|_{p,\Omega} \geq \left( \frac{\alpha_p}{2} |\Omega| - \varepsilon \right) \|\psi\|_{p,\Omega}
\]

and hence

\[
\|G_c L^p(\Omega) \to L^p(c, b)\| \geq \frac{\alpha_p}{2} |\Omega|.
\]

We also have for the above $\Psi$ that

\[
\|\Psi\|_{L^p_c(\Omega)} = \|\Psi\|_{p,\Omega}.
\]

To see this, first note that, for any $f \in L^p(\Omega)$, there exists a unique constant $c_f$ such that

\[
\|f\|_{L^p_c(\Omega)} = \inf_{k \in \mathbb{C}} \|f - k\|_{p,\Omega} = \|f - c_f\|_{p,\Omega};
\]

see [8, Lemma 3.6]. If $k_0 = c\phi$, we have, with $d = \frac{(b-a)}{2}$, since $\Psi(c+t) = -\Psi(c-t)$ for $0 \leq t \leq d$,

\[
\|\Psi - k_0\|_{p,\Omega}^p = \int_0^d |\Psi(c + t) - k_0|^p dt + \int_0^d |\Psi(c - t) - k_0|^p dt
\]
\[
= \int_0^d |-\Psi(c - t) - k_0|^p dt + \int_0^d |-\Psi(c + t) - k_0|^p dt
\]
\[
= \|\Psi + k_0\|_{p,\Omega}^p;
\]

whence $k_0 = 0$ and (2.8). It therefore follows from (2.6) that

\[
\|\Psi\|_{L^p_c(\Omega)} \geq \left( \frac{\alpha_p}{2} |\Omega| - \varepsilon \right) \|\Psi'\|_{p,\Omega}
\]

and so

\[
\|E_C(\Omega)\| \geq \frac{\alpha_p}{2} |\Omega|.
\]

For $f \in W^{1,p}(\Omega)$,

\[
(G_c f')(x) = \int_c^x f'(t) dt = f(x) - f(c).
\]

Hence,

\[
\|f\|_{L^p_c(\Omega)} \leq \|G_c f'\|_{p,\Omega} \leq \|G_c L^p(\Omega) \to L^p(\Omega)\| \|f'\|_{p,\Omega}
\]

gives

\[
\|E_C(\Omega)\| \leq \|G_c L^p(\Omega) \to L^p(\Omega)\|.
\]

Furthermore, by (2.3)

\[
\|G_c f\|_{p,\Omega}^p = \|G_c f\|_{p,(a,c)}^p + \|G_c f\|_{p,(c,b)}^p
\]
\[
\leq \left( \frac{\alpha_p}{2} |\Omega| \right)^p \|f\|_{p,\Omega}^p
\]
and so
\[ \| E_C(\Omega) \| \leq \| G_c L^p(\Omega) \| \rightarrow L^p(\Omega) \| \leq \frac{\alpha_p}{2} |\Omega|. \]
This (2.7) and (2.9) give (2.4) for $E_C(\Omega)$ and $\| G_c \|$.

The proof for $E_0(\Omega)$ is similar, with the above $\Psi$ replaced by
\[ \Psi(x) = \begin{cases} (G_a \phi)(x), & a \leq x \leq c, \\ -(G_b \phi)(x), & c < x \leq b. \end{cases} \]

\[ \square \]

**Remark 2.2** It has recently been proved in [1], [2] and [4] that
\[ \alpha_p = p^{1/p} \frac{p^{1/p'}}{p'} \sin(\pi/p) \] (2.10)
where $p' = p/(p - 1)$.

In the next theorem we use the following notation:
\[ \nu_C(\varepsilon, \Omega) := \max \{ m : a_m(E_C(\Omega)) \geq \varepsilon \} \]
\[ \mu_C(\varepsilon, \Omega) := \max \{ \dim S : \alpha(S) \leq 1/\varepsilon \} \]
where
\[ \alpha(S) = \sup_{f \in S \subset W^{1,p}_C(\Omega)} \frac{\| f' \|_{L^p(\Omega)}}{\| f \|_{L^p(\Omega)}} \]
and $a_m(E_C(\Omega))$ is the $m$th approximation number of $E_C(\Omega)$. Also, $
u_0(\varepsilon, \Omega), \mu_0(\varepsilon, \Omega)$ are the corresponding numbers when $W^{1,p}_C(\Omega)$ is replaced by $W_0^{1,p}(\Omega)$, and thus $E_C(\Omega)$ by $E_0(\Omega)$.

**Theorem 2.3** For $1 < p < \infty$ and $n \in \mathbb{N}$ we have
\[ a_n(E_0(\Omega)) = a_n(E_C(\Omega)) = \frac{\alpha_p}{2n} |\Omega| \] (2.11)
\[ \mu_0(\frac{\alpha_p}{2n} |\Omega|, \Omega) = \nu_0(\frac{\alpha_p}{2n} |\Omega|, \Omega) \]
\[ = \nu_C(\frac{\alpha_p}{2n} |\Omega|, \Omega) = n. \] (2.12)

Hence,
\[ \lim_{\varepsilon \to 0} \varepsilon \mu_0(\varepsilon, \Omega) = \lim_{\varepsilon \to 0} \varepsilon \nu_0(\varepsilon, \Omega) \]
\[ = \lim_{\varepsilon \to 0} \varepsilon \nu_C(\varepsilon, \Omega) = \frac{\alpha_p}{2} |\Omega|. \] (2.14)

**Proof.** Let $\Omega = \bigcup_{i=0}^n \Omega_i$, where $\Omega_0 = (a, a_1), \Omega_i = [a_i, a_{i+1}), i = 1, \ldots, n-1, \Omega_n = [a_n, b)$, and $a_1 - a = b - a_n = \frac{b-a}{2n}, a_{i+1} - a_i = \frac{b-a}{n}, i = 1, 2, \ldots, n-1$. Set $P f(x) = \sum_{i=1}^{n-1} f(c_i) \chi_i(x)$, where $c_i = \frac{a_{i+1} + a_i}{2}$ and
\( \chi_i \) is the characteristic function of \( \Omega_i \). Then, by (2.4), for \( f \in C^1_0(\Omega) \) and with \( a_0 = a, c_n = b, \)

\[
\| E_0(\Omega)f - Pf \|_{p, \Omega}^p = \sum_{i=0}^n \| G_{c_i}f' \|_{p, \Omega_i}^p 
\leq \left\{ \left( \frac{\alpha_p}{2n} \right) |\Omega| \| f' \|_{p, \Omega} \right\}^p.
\]

Hence, since \( C^1_0(\Omega) \) is dense in \( W^{1,p}_0(\Omega) \), we have

\[
\| E_0 - P |W^{1,p}_0(\Omega) \to L^p(\Omega)\| \leq \frac{\alpha_p}{2n} |\Omega|.
\]
(2.15)

Since rank \( P \leq n - 1 \), it follows that

\[
a_n(E_0(\Omega)) \leq \frac{\alpha_p}{2n} |\Omega|
\]
and so

\[
\nu_0(\frac{\alpha_p}{2n} |\Omega|, \Omega) \leq n.
\]
(2.17)

If \( \mathcal{M} \) is the subspace of \( W^{1,p}_0(\Omega) \) spanned by \( \{ \phi_1, \cdots, \phi_n \} \), where, for \( \varepsilon > 0 \), the \( \phi_i \in W^{1,p}_0(\Omega_i) \) are translations of one another, and

\[
\frac{\| \phi'_i \|_{p, \Omega_i}}{\| E_0(\Omega) \phi_i \|_{p, \Omega_i}} \leq \frac{1}{\left( \frac{\alpha_p}{2n} - \varepsilon \right)},
\]
then \( \dim \mathcal{M} = n \) and for non-zero \( \phi \in \mathcal{M}, \)

\[
\frac{\| \phi' \|_{p, \Omega}}{\| E_0(\Omega) \phi \|_{p, \Omega}} \leq \frac{1}{\left( \frac{\alpha_p}{2n} - \varepsilon \right)}.
\]

It follows that \( \mu_0\left( \frac{\alpha_p}{2n} |\Omega|, \Omega \right) \geq n \). Since \( \mu_0(\varepsilon, \Omega) \leq \nu_0(\varepsilon, \Omega) \) is always true by [7, Lemma 5.1, (5.6)], we have proved that

\[
\mu_0\left( \frac{\alpha_p}{2n} |\Omega|, \Omega \right) = \nu_0\left( \frac{\alpha_p}{2n} |\Omega|, \Omega \right) = n.
\]

and therefore \( a_n(E_0(\Omega)) = \frac{\alpha_p}{2n} |\Omega| \).

To prove the result for \( E_\mathcal{C}(\Omega) \), we choose the partition \( \Omega = \bigcup_{i=1}^n \Omega_i \), where \( |\Omega_i| = \frac{|\Omega|}{n} \) for \( i = 1, \cdots, n \). For \( g \in [f], f \in W^{1,p}_C(\Omega) \), set \( P_g(x) = \Sigma_{i=1}^n f(c_i) \chi_i(x), \) where \( c_i \) is the mid-point of \( \Omega_i \). As a map from \( W^{1,p}_C(\Omega) \) into \( L^p_\mathcal{C}(\Omega) \), \( P \) has rank \( \leq n - 1 \) as is seen on writing \( P_g = f(c_1) + \Sigma_{i=2}^n f(c_i)(\chi_i - \chi_1) \). Furthermore,

\[
\| f - P_g \|_{p, \Omega} = \Sigma_{i=1}^n \| G_{c_i}f' \|_{p, \Omega_i} 
\leq \left\{ \left( \frac{\alpha_p}{2n} \right) |\Omega| \| f' \|_{p, \Omega} \right\}^p.
\]

Hence

\[
\| E_\mathcal{C}(\Omega) - P |W^{1,p}_C(\Omega) \to L^p_\mathcal{C}(\Omega)\| \leq \frac{\alpha_p}{2n} |\Omega|
\]
and so, since rank \( P \leq n - 1 \),

\[
a_n(E_\mathcal{C}(\Omega)) \leq \frac{\alpha_p}{2n} |\Omega|.
\]
From Theorem 2.2, for \( i = 1, \ldots, n \),
\[
\| E_C(\Omega_i)W_{1,p}^i(\Omega_i) \| = \sup \{ \inf_{c \in C} \| f - c \|_{p,\Omega_i} : \| f' \|_{p,\Omega_i} = 1 \} = \frac{\alpha_p}{2n} |\Omega|.
\]
Thus, if \( \delta < \frac{\alpha_p}{2n} |\Omega| \), there exists \( \psi_i \in W_{1,p}^i(\Omega_i) \) such that \( \| \psi_i \|_{p,\Omega_i} > 0 \) and
\[
\| \psi_i \|_{L_p^C(\Omega_i)} \equiv \{ \inf_{c \in C} \| f - c \|_{p,\Omega_i} \} > \delta \| \psi_i' \|_{p,\Omega_i}.
\]
Define \( \psi'_i \) to be zero outside \( \Omega_i \) and such that \( \psi_i \in W_{1,p}^i(\Omega_i) \). Now let \( P : W_{1,p}^1(\Omega) \to L_p^p(\Omega) \) have rank \( < n \). It follows that there exist constants \( \lambda_i, i = 1, \ldots, n \), not all zero, such that
\[
[\psi] = \sum_{i=1}^n \lambda_i [\psi_i], \quad P[\psi] = 0.
\]
Then
\[
\|(E_C(\Omega) - P)[\psi]\|_{L_p^C(\Omega)} \geq \delta \sum_{i=1}^n \lambda_i \| \psi_i' \|_{p,\Omega_i} = \delta \sum_{i=1}^n \lambda_i \| \psi_i' \|_{p,\Omega_i}.
\]
Since \( P \) of rank \( < n \) and \( \delta < \frac{\alpha_p}{2n} |\Omega| \) are arbitrary, we have proved that
\[
a_n(E_C(\Omega)) \geq \frac{\alpha_p}{2n} |\Omega|,
\]
and hence the proof is complete. \( \square \)

3. Quotient spaces, norms and approximation numbers

The domain \( \Omega \) is now a subset of \( \mathbb{R}^n, n \geq 1 \) of finite volume \( |\Omega| \). The quotient space \( L_C(\Omega) \equiv L^p(\Omega)/C \) has norm
\[
\| f \|_{L_C(\Omega)} \equiv \|[f] \|_{p,\Omega} := \inf_{c \in C} \| f - c \|_{p,\Omega}
\]
and functions are equivalent if and only if they differ by a constant a.e. Another relevant space in subsequent analysis is
\[
L_g^p(\Omega) := \{ f \in L^p(\Omega) : \Omega_g(f) := \int_{\Omega} f g dx = 0 \}
\]
where \( g \in L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1, \) and \( \int_{\Omega} g dx \neq 0 \). Note that \( L_g^p(\Omega) \) is the space denoted by \( L_M^N(\Omega) \) in [7, Section 3]. Clearly \( L_g^p(\Omega) \) is a closed subspace of \( L^p(\Omega) \) and \( f \mapsto f - \Omega_g(f) \) is a projection of \( L^p(\Omega) \) onto \( L_g^p(\Omega) \). We shall write
\[
\| f \|_{L_g^p(\Omega)} = \| f - \Omega_g(f) \|_{p,\Omega}, \quad f \in L^p(\Omega)
\]
and hence
\[
\| f \|_{L_g^p(\Omega)} = \| f \|_{p,\Omega}, \quad f \in L_g^p(\Omega).
\]
Lemma 3.1 Let $V_\Omega$ denote the canonical map $V_\Omega : L^p(\Omega) \rightarrow L^p_\mathbb{C}(\Omega)$, given by $f \mapsto [f]$. Then, for all $f \in L^p(\Omega)$,
\[
\|V_\Omega f\|_{L^p_\mathbb{C}(\Omega)} \leq \|f\|_{L^p(\Omega)} \leq c(p, g)\|V_\Omega f\|_{L^p_\mathbb{C}(\Omega)},
\]
where
\[
c(p, g) = 1 + \frac{|\Omega|^{1/p}\|g\|_{p',\Omega}}{\int_\Omega g dx}.
\]
When $p = 2$ and $g = 1$
\[
\|V_\Omega f\|_{L^2_\mathbb{C}(\Omega)} = \|f\|_{L^2_\mathbb{C}(\Omega)} = \|f - \Omega_1(f)\|_{2,\Omega}.
\]

Proof. The first inequality in (3.4) is obvious, while the second is a consequence of
\[
|\Omega_g(f)| \leq \frac{1}{\int_\Omega g(x) dx} \|g\|_{p',\Omega} \|f\|_{p,\Omega}
\]
which follows from Hölder’s inequality. The identity (3.5) is well known. □

We now set
\[
W^{1,p}_g(\Omega) := \{f : f \in W^{1,p}(\Omega), \Omega_g(f) = 0\}
\]
with norm
\[
\|f\|_{W^{1,p}_g(\Omega)} := \|\nabla f\|_{p,\Omega}
\]
and
\[
E_g(\Omega) : W^{1,p}_g(\Omega) \hookrightarrow L^p_\mathbb{C}(\Omega);
\]
$W^{1,p}_1(\Omega)$ is the space $W^{1,p}_M(\Omega)$ in [7]. Note that $W^{1,p}_g(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$ if the Poincaré inequality (1.4) is satisfied. We have the direct sum decomposition
\[
W^{1,p}(\Omega) = W^{1,p}_g(\Omega) + \mathbb{C}
\]
determined by
\[
f = [f - \Omega_g(f)] + \Omega_g(f)
\]
since $\Omega_g(\Omega_g(f)) = \Omega_g(f)$, and the map $W : [f] \mapsto f - \Omega_g(f)$ is an isometry between $W^{1,p}_C(\Omega)$ and $W^{1,p}_g(\Omega)$. The map $W$ is also a topological isomorphism of $L^p_\mathbb{C}(\Omega)$ onto $L^p_\mathbb{C}(\Omega)$, by Lemma 3.1, satisfying
\[
\|f\|_{L^p_\mathbb{C}(\Omega)} \leq \|W[f]\|_{L^p_\mathbb{C}(\Omega)} \leq c(p, g)\|f\|_{L^p_\mathbb{C}(\Omega)},
\]
Note that if $f \in W^{1,p}_g(\Omega)$ and $h \in [f]$, then $h = f + \Omega_g(h)$ and similarly between $L^p_\mathbb{C}(\Omega)$ and $L^p_\mathbb{C}(\Omega)$, with $W^{-1}f = [f]$. In the analysis to follow, it is helpful to distinguish between the maps
\[
W_1 : [f] \mapsto f - \Omega_g(f) : W^{1,p}_C(\Omega) \rightarrow W^{1,p}_g(\Omega)
\]
\[
W_0 : [f] \mapsto f - \Omega_g(f) : L^p_\mathbb{C}(\Omega) \rightarrow L^p_\mathbb{C}(\Omega).
\]
From the preceding remarks, both maps are surjective, \( W_1 \) is an isometry and
\[
1 \leq \|W_0\| \leq c(p, g).
\] (3.12)

It is readily seen that
\[
E_g(\Omega) = W_0E_C(\Omega)W_1^{-1} \quad (3.13)
\]
\[
E_C(\Omega) = W_0^{-1}E_g(\Omega)W_1. \quad (3.14)
\]

**Theorem 3.2** For all \( f \in L^p(\Omega) \),
\[
\|V_\Omega f\|_{L^p_0(\Omega)} = \min_{g \in S} \|f - \Omega_g(f)\|_{p, \Omega}, \quad (3.15)
\]
where
\[
S := \{g : g \in L^p(\Omega), \int_\Omega g(x)dx \neq 0\}.
\]

When \( p = 2 \) we have
\[
\|V_\Omega f\|_{L^2_0(\Omega)} = \|f - \Omega_1(f)\|_{2, \Omega}.
\]

**Proof.** If \( f_0 \in L^p(\Omega) \) and \( V_\Omega f_0 = 0 \), then \( f_0 = c \), a constant a.e., and, for any \( g \in S \),
\[
\|V_\Omega f_0\|_{L^p_0(\Omega)} = \|c - \Omega_g(c)\|_{p, \Omega} = 0.
\]
Hence, from (3.4),
\[
\|V_\Omega f_0\|_{L^p_0(\Omega)} = \min_S \|f_0 - \Omega_g(f_0)\|_{p, \Omega}.
\]
Suppose \( \|V_\Omega f\|_{L^p_0(\Omega)} \neq 0 \). Then, there exists a unique constant \( c = c(f) \) such that
\[
\|V_\Omega f\|_{L^p_0(\Omega)} = \|f - c\|_{p, \Omega}
\]
(c.f. [8, Lemma 3.6]). Since \( f - c \neq 0 \), the one-dimensional vector space \( \{\lambda(f - c) : \lambda \in \mathbb{C}\} \) does not contain the constant function \( 1 \). Therefore, by the Hahn-Banach Theorem, there exists \( l \in (L^p(\Omega))^* \), the dual of \( L^p(\Omega) \), such that
\[
l(f - c) = 0, \quad l(1) = 1.
\]
Hence, there exists \( g_0 \in L^p(\Omega) \) such that
\[
\int_\Omega [f(x) - c]g_0(x)dx = 0 \quad \int_\Omega g_0(x)dx = 1.
\]
Hence, \( \Omega_{g_0}(f) = c \) and
\[
\|V_\Omega f\|_{L^p_0(\Omega)} = \|f - \Omega_{g_0}(f)\|_{p, \Omega}.
\]
Since
\[ \|V_\Omega f\|_{L^p_\Omega(\Omega)} \leq \|f - \Omega_g(f)\|_{p,\Omega}, \]
for any \( g \in \mathcal{S} \), (3.15) follows. The case \( p = 2, g = 1 \) was noted in (3.5).

The approximation numbers of \( E_g(\Omega) \) and \( E_C(\Omega) \) both feature in the analysis to follow. They are related by

**Lemma 3.3** For any \( g \in \mathcal{S} \),
\[ a_m(E_C(\Omega)) \leq a_m(E_g(\Omega)) \leq c(p, g)a_m(E_C(\Omega)) \tag{3.16} \]
where \( c(p, g) \) is given in Lemma 3.1; when \( g = 1 \), \( c(2, 1) = 1 \) and \( c(p, 1) = 2 \) for \( p \neq 2 \). Therefore
\[ \nu_C(\varepsilon, \Omega) \leq \nu_g(\varepsilon, \Omega) \leq \nu_C\left(\frac{\varepsilon}{c(p, g)}, \Omega\right). \tag{3.17} \]

**Proof.** Let \( P : W^{1,p}_C(\Omega) \to L^p_C(\Omega) \) be of rank \( < m \). Then, from (3.13) and (3.14),
\[ \|E_g(\Omega) - W_0PW^{-1}_1|W^{1,p}_g(\Omega) \to L^p_g(\Omega)\|
= \|W_0(E_C - P)W^{-1}_1|W^{1,p}_g(\Omega) \to L^p_g(\Omega)\|
\leq c(p, g)\|E_C - P|W^{1,p}_C(\Omega) \to L^p_C(\Omega)\|. \]
Since \( \text{rank}(W_0PW^{-1}_1) \leq \text{rank}P < m \), the second inequality in (3.16) follows. The first inequality is proved similarly. The inequalities (3.17) are immediate consequences of (3.15). \( \square \)

Similar results to the preceding ones in this section hold also on the tree \( \Gamma \). The analogous spaces defined on \( \Gamma \) are the following:
\[ L^p_C(\Gamma, d\mu) : \|u\|_{L^p_C(\Gamma, d\mu)} := \inf_{c \in C} \|u - c\|_{L^p_C(\Gamma, d\mu)}; \tag{3.18} \]
\[ L^p_h(\Gamma, d\mu) : = \{ u \in L^p(\Gamma, d\mu) : \omega_h(u) := \frac{\int_{\Gamma} uhd\mu}{\int_{\Gamma} hd\mu} = 0 \} \]
\[ \|u\|_{L^p_h(\Gamma, d\mu)} : = \|u - \omega_h(u)\|_{L^p(\Gamma, d\mu)}; \tag{3.19} \]
where \( h \in L^p(\Gamma, d\mu) \) and \( \int_{\Gamma} hd\mu \neq 0 \);
\[ L^{1,p}_C(\Gamma, d\mu) : = \|u\|_{L^{1,p}_C(\Gamma, d\mu)} = \|u'\|_{L^p(\Gamma, d\mu)}; \tag{3.20} \]
\[ L^{1,p}_h(\Gamma, d\mu) : = \{ u \in L^{1,p}(\Gamma, d\mu) : \omega_h(u) = 0 \} \]
\[ \|u\|_{L^{1,p}_h(\Gamma, d\mu)} : = \|u'\|_{L^p(\Gamma, d\mu)}. \tag{3.21} \]

We denote the associated embeddings by
\[ I_C(\Gamma) : \; L^{1,p}_C(\Gamma, d\mu) \to L^p_C(\Gamma, d\mu); \tag{3.22} \]
\[ I_h(\Gamma) : \; L^{1,p}_h(\Gamma, d\mu) \to L^p_h(\Gamma, d\mu). \tag{3.23} \]
The maps
\[
T_1 : [u] \mapsto u - \omega_h(u) : L^{1,p}_C(\Gamma, d\mu) \to L^{1,p}_h(\Gamma, d\mu) \quad (3.24)
\]
\[
T_0 : [u] \mapsto u - \omega_h(u) : L^p_C(\Gamma, d\mu) \to L^p_h(\Gamma, d\mu) \quad (3.25)
\]
are respectively an isometry and an isomorphism, with
\[
1 \leq \|T_0\| \leq d(p, h) := 1 + \frac{\mu(\Gamma)^{1/p}\|h\|_{L^p'(\Gamma, d\mu)}}{|\int_{\Gamma} h d\mu|}. \quad (3.26)
\]
The analogue of Theorem 3.3 is

**Theorem 3.4**
\[
a_m(I_C(\Gamma)) \leq a_m(I_h(\Gamma)) \leq d(p, h)a_m(I_C(\Gamma)) \quad (3.27)
\]
and
\[
\nu_C(\varepsilon, \Gamma) \leq \nu_h(\varepsilon, \Gamma) \leq \nu_C \left( \varepsilon \frac{d(p, h)}{d(p, h)}, \Gamma \right), \quad (3.28)
\]
where \(d(p, h)\) is given in (3.26); in particular \(d(2, 1) = 1\) and \(d(p, 1) = 2\) for \(p \neq 2\).

### 4. An upper bound for \(\nu_1(\varepsilon, \Omega)\)

**Theorem 4.1** Given \(\varepsilon > 0\), let \(q(\varepsilon), h(\varepsilon) \in \mathbb{N}\) be such that

\[
\Omega = \bigcup_{i=0}^{q(\varepsilon)} \Omega_i(\varepsilon) \bigcup \bigcup_{i=q(\varepsilon)+1}^{q(\varepsilon)+h(\varepsilon)} \Omega_i(\varepsilon) \bigcup \mathcal{N}, \quad (4.1)
\]

where the \(\Omega_i(\varepsilon)\) are disjoint open sets, \(\mathcal{N}\) is a null set and

- \(\Omega_0(\varepsilon) = \bigcup_{j=1}^{r(\varepsilon)} Q_j(\varepsilon) \cup \mathcal{N}\), where the \(Q_j(\varepsilon)\) are disjoint rectangular blocks with edge lengths \(l_1(\varepsilon, j) \leq l_2(\varepsilon, j) \cdots \leq l_n(\varepsilon, j)\);
- \(\Omega_i(\varepsilon), i = 1, \cdots, q(\varepsilon)\) are generalised ridged domains with associated trees \(\Gamma_i(\varepsilon)\), for which the constants \(\alpha, \zeta\) in Definition 1.1 are such that \(\alpha, \zeta^{-1}\) are bounded independently of \(\varepsilon\) and

\[
\sup_{x \in \Omega_i(\varepsilon)} (\varrho \circ \tau)(x) < \varepsilon,
\]

(see [6,7]);
- \(\Omega_i(\varepsilon), i = q(\varepsilon) + 1, \cdots, q(\varepsilon) + h(\varepsilon)\) have the Poincaré property

\[
\|[f]\|_{p,\Omega_i(\varepsilon)} = \inf_{c \in \mathbb{C}} \|f - c\|_{p,\Omega_i(\varepsilon)} \leq \varepsilon\|\nabla f\|_{p,\Omega_i(\varepsilon)}, \quad f \in W^{1,p}(\Omega_i(\varepsilon)).
\]

For any \(\delta \in (0, 1)\) and \(k = 0, 1, \cdots, n - 1\), define

\[
L_k := \{ j : j \in \{1, \cdots, r(\varepsilon)\}, l_k(\varepsilon, j) \leq \varepsilon\delta^{-1} < l_{k+1}(\varepsilon, j) \} \quad (4.2)
\]
\[ p_j(\varepsilon, \delta, 0) := \Pi_{s=1}^n (\epsilon^{-1} \delta l_s(\varepsilon, j) + 1), \]
\[ - \Pi_{s=1}^n (\epsilon^{-1} \delta l_s(\varepsilon, j)), \]
\[ p_j(\varepsilon, \delta, k) := [\epsilon^{-1} \delta]^{n-k}\Pi_{s=k+1}^n l_s(\varepsilon, \delta), 1 \leq k \leq n-2, \]
\[ p_j(\varepsilon, \delta, n-1) := 1, \]  

where \( l_0(\varepsilon, j) = 0. \)

Then
\[ \varepsilon^n [\nu_1(\varepsilon, \Omega) + 1] \leq \sum_{j \in \mathcal{L}} \nu_1(\varepsilon, \Omega)(\varepsilon) + \sum_{i=1}^{q(\varepsilon)} \nu_1(\varepsilon, \Omega_i(\varepsilon)) + 1 \]
\[ + \sum_{i=q(\varepsilon)+1}^{q(\varepsilon)+h(\varepsilon)} \nu_1(\varepsilon, \Omega_i(\varepsilon)) + 1 \]
\[ = I_1 + I_2 + I_3 \]  

say. Let \( Q \) be a rectangular block with edges \( l_1 \leq l_2 \leq \cdots \leq l_n. \) Suppose first that
\[ 1 \leq \epsilon^{-1} \delta l_1 \leq \epsilon^{-1} \delta l_2 \leq \cdots \epsilon^{-1} \delta l_n, \]  

and set \( M_r = [\epsilon^{-1} \delta l_r]. \) Then, \( M_r \geq 1 \) and, for some \( \theta_r \in [0,1), \) we have
\[ \epsilon^{-1} \delta l_r = M_r + \theta_r. \]  

We divide \( \epsilon^{-1} Q \) into \( M_1M_2 \cdots M_n \) n-dimensional cubes of side \( \mu = \delta^{-1}, \) and a remainder \( S \) made up of \( \Pi_{s=1}^n (M_s+1) - \Pi_{s=1}^n M_s \) rectangular blocks of sides \( \eta_1\mu, \cdots , \eta_n\mu, \) where \( \eta_r = 1 \) or \( \theta_r. \) Further, subdivide each of the rectangular blocks in \( S \) into \( k(\mu) \) rectangular blocks \( T \) which are sufficiently small that \( \nu_1(1, T) = 0; \) note that \( k(\mu) = \mathcal{O}(\mu^n). \) It
follows from [7, Lemmas 5.3 and 5.6] that
\[ \nu_1(\varepsilon, Q) + 1 = \nu_1(1, \varepsilon^{-1}Q) + 1 \]
\[ \leq M_1 \cdots M_n \{ \nu_1(1, \mu U) + 1 \} \]
\[ + \left( \Pi^n_{s=1} (M_s + 1) - \Pi^n_{s=1} M_s \right) k(\mu) \]
\[ \leq \varepsilon^{-n} \delta^n |Q| \{ [\nu_1(1, \mu U) + 1] \}
\]
\[ + K\delta^{-n} \left\{ \Pi^n_{s=1} (\varepsilon^{-1} \delta_{s+1} + 1) - \Pi^n_{s=1} (\varepsilon^{-1} \delta_{l_s}) \right\}. \quad (4.9) \]

Next, suppose that \[ \varepsilon^{-1} \delta_{l_k} < 1 \leq \varepsilon^{-1} \delta_{l_{k+1}} \] (4.10)
for some \( k \in (0, n-1) \). We then have that \( M_1 = \cdots = M_k = 0 \) and \( 1 \leq M_{k+1} \leq \cdots \leq M_n \). This time we divide \( \varepsilon^{-1}Q \) into at most \( \Pi^n_{s=k+1} (M_s + 1) \) rectangular blocks whose sides are of length at most \( \mu \).

As in the previous case, each of these rectangular blocks are divided into \( \bigcirc(\mu^n) \) blocks \( T \) which are sufficiently small that \( \nu_1(1, T) = 0 \). We now have
\[ \nu_1(\varepsilon, Q) + 1 = \nu_1(1, \varepsilon^{-1}Q) + 1 \]
\[ \leq K \Pi^n_{s=k+1} (M_s + 1) \mu^n \]
\[ \leq K \delta^{-n} \Pi^n_{s=k+1} (\varepsilon^{-1} \delta_{l_s}). \quad (4.11) \]

Finally, suppose that \[ \varepsilon^{-1} \delta_{l_n} < 1. \] (4.12)
We again divide \( \varepsilon^{-1}Q \) into \( \bigcirc(\mu^n) \) blocks \( T \) for which \( \nu_1(1, T) = 0 \) and obtain
\[ \nu_1(\varepsilon, Q) + 1 = \nu_1(1, \varepsilon^{-1}Q) + 1 \]
\[ \leq K \delta^{-n}. \quad (4.13) \]

From (4.9), (4.11) and (4.13) we have
\[ \varepsilon^n I_1 = \varepsilon^n \{ \nu_1(\varepsilon, \Omega_0(\varepsilon)) + 1 \} \leq \sum_{j=1}^{r(\varepsilon)} \varepsilon^n \{ \nu_1(\varepsilon, Q_j(\varepsilon)) + 1 \} \]
\[ \leq \delta^n [\nu_1(\delta, U) + 1] \sum_{j \in L_0} |Q_j| \]
\[ + K \varepsilon^n \delta^{-n} \sum_{k=0}^{n-1} \sum_{j \in L_k} p_j(\varepsilon, \delta, k). \quad (4.14) \]

Let \( \Omega' \) be any GRD with associated tree \( \Gamma' \), and, in the notation of [5,7], set
\[ (TF)(x) := (F \circ \tau)(x), \quad x \in \Omega', \quad (4.15) \]
\[ (Mf)(t) := \frac{1}{|B_t|} \int_{B_t} f(x) dx, \quad (4.16) \]
where $B_t = B(u(t), \rho(t)), t \in \Gamma'$, and

\[
\psi(s) := \frac{ds}{d\mu}, \quad u(s) := \psi(s)^{1/p}, \quad v(s) := \psi(s)^{-1/p}. \tag{4.17}
\]

where $ds$ denotes Lebesgue measure on $\Gamma'$. The maps

\[
\begin{align*}
T_1 & : [F] \mapsto [TF] : L^1_{C}(\Gamma', d\mu) \to W^{1,p}_{C}(\Omega') \\
T_0 & : [F] \mapsto [TF] : L^p_{C}(\Gamma', d\mu) \to L^p_{C}(\Omega') \\
M_1 & : [f] \mapsto [Mf] : W^{1,p}_{C}(\Omega') \to L^1_{C}(\Gamma', d\mu) \\
M_0 & : [f] \mapsto [Mf] : L^p_{C}(\Omega') \to L^p_{C}(\Gamma', d\mu) 
\end{align*}
\tag{4.18}
\]

are well-defined, and we have the commuting diagram
\[ W^{1,p}_C(\Omega') \xrightarrow{T_1} L^{1,p}_C(\Gamma', d\mu) \]
\[ L^p_C(\Omega') \xrightarrow{T_0} L^p_C(\Gamma', d\mu) \] 

(4.19)

The following identities have important implications for the proof:

\[ M_0 E_C = I_C M_1 \]  
\[ E_C T_1 = T_0 I_C \]  
(4.20)  
(4.21)

we suppress the dependence on \( \Omega' \) and \( \Gamma' \) to simplify the notation.

From

\[ E_C = E_C - T_0 M_0 E_C + T_0 M_0 E_C \]
we have by [3, Proposition II.2.2]

\[ a_m(E_C) \leq \|E_C - T_0 M_0 E_C\| + a_m(T_0 M_0 E_C) \]

and, on using (4.20),

\[ a_m(T_0 M_0 E_C) = a_m(T_0 I_C M_1) \leq \|T_0\| a_m(I_C M_1) \leq a_m(I_C)\|M_1\|. \]

Furthermore, from [7, Lemma 3.4],

\[ \|E_C - T_0 M_0 E_C\| \leq c(\Omega') \sup_{\Omega'} (\rho \circ \tau)(x) =: c(\Omega')k(\Omega') \]

we have that

\[ c(\Omega') \leq \zeta^{-1}(\alpha + 1)^n c_p \{ \alpha + 1 + 2^{n+1} c_p \} \]

where \( \zeta, \alpha \) are the constants in Definition 1.1, and \( c_p \) is the norm of the maximal function as a map on \( L^p(\mathbb{R}^n) \). In view of the hypothesis, when \( \Omega' = \Omega_i(\epsilon), i = 1, 2, \ldots, q(\epsilon) \), the associated constants \( c(\Omega') \) are bounded independently of \( \epsilon \). Hence,

\[ a_m(E_C) \leq c(\Omega') + \|M_1\| a_m(I_C). \]

(4.22)

It follows for \( \Omega' = \Omega_i(\epsilon), i = 1, 2, \ldots, q(\epsilon) \) that

\[ a_m(I_C(\Gamma')) < \epsilon \Rightarrow a_m(E_C(\Omega')) < [c + \|M_1^{(i)}\|\epsilon] \]

(4.23)

and hence

\[ \nu_C(\epsilon, \Gamma_i) \geq \nu_C([c^{(i)} + \|M_1^{(i)}\|\epsilon, \Omega_i(\epsilon))]_{i = 1, 2, \ldots, q(\epsilon)}, \]

(4.24)
where $M_1^{(i)}$ is the map $M_1$ in (4.18) when $\Omega', \Gamma'$ are $\Omega_i(\varepsilon), \Gamma_i(\varepsilon)$ respectively; from [5, Lemma 4.2] and [7, Lemma 3.3], $\|M_1^{(i)}\| \leq c_p(\alpha + 1)^n$ which is bounded independently of $\varepsilon$ by hypothesis. In view of Lemma 3.3 and Theorem 3.4, (4.24) yields
\begin{equation}
\nu_1(\varepsilon, \Gamma_i(\varepsilon)) \geq \nu_1(c(p, 1)[c^{(i)} + \|M_1^{(i)}\|] \varepsilon, \Omega_i(\varepsilon)),
\end{equation}
and in (4.6),
\begin{equation}
\varepsilon^n I_2 \leq \sum_{i=1}^{q(\varepsilon)} \varepsilon^n \{\nu_1(c_i \varepsilon, \Gamma_i(\varepsilon)) + 1\}
\end{equation}
where $c_i = (c(p, 1)[c^{(i)} + \|M_1^{(i)}\|])^{-1}$.

For the $\Omega_i(\varepsilon)$ in $I_3$, we have that
\begin{equation}
\|E_{C_1}W_C^{1,p}(\Omega_i(\varepsilon) \to L_C^p(\Omega_i(\varepsilon))\| \leq \varepsilon
\end{equation}
and hence
\begin{equation}
\nu_1(\varepsilon, \Omega_i(\varepsilon)) = 0, \quad i = q(\varepsilon) + 1, \cdots, q(\varepsilon) + h(\varepsilon).
\end{equation}
The existence of the limit $L_U$ and the identity (4.5) are proved in [7, Theorem 5.7]. The theorem is therefore proved. □

**Remark 4.2** In the hypothesis of Theorem 4.1, $\Omega$ is predominantly composed of the set $\Omega_0(\varepsilon)$ which is a union of rectangular blocks, the remainder being made up of $q(\varepsilon)$ GRDs of "small width" and $h(\varepsilon)$ sets satisfying the Poincaré inequality and having "small diameter". The GRDs are irregular portions of $\Omega$ (like cusps or spirals) left over after $\Omega_0(\varepsilon)$ has been removed, while the Poincaré domains are less singular. If we choose $\delta = \delta(\varepsilon)$, say, in (4.4), the first term on the right-hand side is the main term as $\varepsilon \to 0$, except in pathological cases when the other terms might dominate and yield a non-standard asymptotic estimate even in the case $p = 2$ for the counting function of the eigenvalues of the Neumann Laplacian. In all cases an appropriate choice of $\delta$ is crucial.

**Remark 4.3** In the estimate (4.4), the limit $L_U$ and the error $\delta^n[\nu_1(\delta, U) + 1] - L_U$ are not known for $p \neq 2$. When $p = 2$,
\begin{equation}
\nu_1(\delta, U) = \nu_{C_1}(\delta, U) = N(\delta^{-n}, \Delta_{N,U}) = (2\pi)^{-n} \omega_n \delta^{-n} + O(\delta^{-n+1})
\end{equation}
as $\delta \to 0$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$; see [7, (5.12)]. When $n = 1$, it follows from (2.14) that
\begin{equation}
\lim_{t \to 0} t \nu_{C_1}(t, U) = \frac{\alpha_p}{2} \quad \text{and hence, from (3.17),}
\end{equation}
\begin{equation}
\frac{\alpha_p}{2} \leq L_U \leq c(p, 1) \frac{\alpha_p}{2},
\end{equation}
where $c(2,1) = 1$ and $c(p,1) = 2$ for $p \neq 2$.

**Remark 4.4** In Theorem 4.2, the rectangular blocks $Q_j(\varepsilon), j = 1,2,\cdots,r(\varepsilon)$ in $\Omega_0(\varepsilon)$ can be replaced by any similar sets which fit together to form a tessellation of $\Omega_0(\varepsilon)$. This observation can greatly ease the application of the result. For instance, to apply our methods and results to the Koch snowflake, the choice of equilateral triangles, or regular hexagons, is appropriate. The proof of Theorem 4.1 goes through as before on noticing that Theorem 5.7 in [7] continues to hold, *mutatis mutandis*, for a regular polyhedron $Q$, namely

$$
\lim_{\varepsilon \to 0} \{\varepsilon^2 \nu_1(\varepsilon, Q)\} = \inf_{\varepsilon > 0} \varepsilon^2[\nu_1(\varepsilon, Q) + 1].
$$

(4.27)

Recent research has uncovered a great deal of information about the quantity $\nu_1(\varepsilon, \Gamma)$ which features on the right-hand side of (4.4). Before we can comment further, we need to establish the connection with the Hardy operator $H_c : L^p(\Gamma) \to L^p(\Gamma)$ on a tree $\Gamma$ defined by

$$
(H_c F)(t) := v(t) \int_c^t F(s) u(s) ds, \quad c, t \in \Gamma.
$$

(4.28)

For $F \in L^{1,p}(\Gamma, d\mu)$ we have

$$
F(t) - F(c) = \int_c^t F'(s) ds
= \int_c^t F'(s) v(s) u(s) ds
$$

by (4.17). This can be written as

$$
F(t) - F(c) = (R_c DF)(t),
$$

(4.29)

where

$$
D : [F] \mapsto F' v : L^{1,p}_c(\Gamma, d\mu) \to L^p(\Gamma)
$$

(4.30)

and

$$
(R_c F)(t) = \int_c^t F(s) u(s) ds; \quad R_c : L^p(\Gamma) \to L^p(\Gamma, d\mu).
$$

(4.31)

Note that $D$ is surjective, and is isometric since

$$
\|DF\|_{p,\Gamma} = \|F'\|_{L^p(\Gamma, d\mu)} = \|F\|_{L^p_c(\Gamma, d\mu)}.
$$

(4.32)

Also, (4.29) gives

$$
I_c(\Gamma) = V \Gamma R_c D = V \Gamma v^{-1} H_c D,
$$

(4.33)

where $V \Gamma : L^p(\Gamma, d\mu) \to L^p_c(\Gamma, d\mu)$ is the canonical map and $v^{-1}$ is multiplication by $1/v$; the map $F \mapsto v^{-1} F$ is an isometry between $L^p(\Gamma)$ and $L^p(\Gamma, d\mu)$. The identity (4.33) leads to the following estimate for the approximation numbers of $I_c(\Gamma)$ in terms of those for $H_c$. 

Lemma 4.5 Let \( H_c \) be the Hardy operator defined in (4.28) with \( u, v \) given in (4.17), and so \( uv = 1 \). Then

\[
\frac{1}{2} a_{m+1}(H_c) \leq a_m(I_C(\Gamma)) \leq a_m(H_c), \quad m \in \mathbb{N}_0
\]  

(4.34)

and hence

\[
\nu(2\epsilon, H_c) \leq \nu_C(\epsilon, \Gamma) \leq \nu(\epsilon, H_c)
\]  

(4.35)

where \( \nu(\epsilon, H_c) := \max\{m \in \mathbb{N}_0 : a_m(H_c) \geq \epsilon\} \).

Proof. From (4.33) and [3, Proposition II.2.2(iii)], we have

\[
a_m(I_C(\Gamma)) \leq \|V_{\Gamma}v^{-1}\| \|D\| a_m(H_c)
\]

\[
\leq a_m(H_c)
\]

since \( v^{-1} \) and \( D \) are isometries and \( \|V_{\Gamma}\| \leq 1 \). To derive the first inequality in (4.34), define \( A : L^p(\Gamma, d\mu) \to \mathbb{C} \) to be the one-dimensional operator

\[
Af := \frac{1}{\mu(\Gamma)} \int f d\mu.
\]

Denote the kernel of \( A \) by \( N \) and the restriction of \( V_{\Gamma} \) to \( N \) by \( V_N \). Then \( V_N \) is one-one, since \( V_N f = V_N g \) implies that \( [f - g] = 0 \) and hence \( f - g = c \), a constant. But this means that \( c = A(f - g) = 0 \) and so \( f = g \). Furthermore \( V_N \) has range \( L^p_C(\Gamma) \), for \( [f] = [f - Af] \) and \( f - Af \in N \). Thus

\[
V_N(I - A)f = [f] = V_{\Gamma}f,
\]

(4.36)

where \( I \) is the identity on \( L^p(\Gamma) \). This gives in (4.33)

\[
a_m(I_C(\Gamma)) = a_m(V_N(I - A)R_c D), \quad R_c = v^{-1} H_c
\]

\[
= \inf \left\{ \|V_N(I - A)R_c D - P|L^p_C(\Gamma, d\mu) \to L^p_C(\Gamma, d\mu)\| : \text{rank } P < m \right\}
\]

\[
= \inf \left\{ \|V_N(R_c - AR_c - V_N^{-1}PD^{-1})|L^p(\Gamma, d\mu) \to L^p_C(\Gamma, d\mu)\| : \text{rank } P < m \right\}
\]

since \( D \) is an isometry of \( L^p_C(\Gamma, d\mu) \) onto \( L^p(\Gamma) \). From (4.36),

\[
\|V_N^{-1}[f]\|_{L^p(\Gamma, d\mu)} \leq 2\|\{f\}\|_{L^p(\Gamma, d\mu)}
\]

and so

\[
\|V_N^{-1}|L^p_C(\Gamma, d\mu) \to L^p(\Gamma, d\mu)\| \leq 2;
\]

hence \( AR_c + V_N^{-1}PD : L^p(\Gamma) \to L^p(\Gamma, d\mu) \) is bounded and of rank \( \leq \text{rank } P + 1 < m + 1 \). It follows that

\[
a_m(I_C(\Gamma)) \geq \|V_N^{-1}\|\inf \left\{ \|R_c - Q|Q : L^p(\Gamma) \to L^p(\Gamma, d\mu)\| : \text{rank } Q < m + 1 \right\}
\]

\[
\geq \frac{1}{2}a_{m+1}(R_c) = \frac{1}{2}a_{m+1}(H_c)
\]
since \( v^{-1} : L^p(\Gamma, d\mu) \to L^p(\Gamma) \) is an isometry. The lemma is therefore proved.

**Remark 4.6** In the case when \( \Gamma \) is an interval \((c, \infty)\), \( H_c \) is bounded if Muckenhausen’s \( A_p \)-condition holds, namely
\[
\sup_{c < t < \infty} \left\{ \left( \int_c^t |u(s)|^{p'} \, ds \right)^{1/p'} \left( \int_t^\infty |v(s)|^p \, ds \right)^{1/p} \right\} < \infty,
\]
i.e. in the notation (4.6)
\[
\sup_{c < t < \infty} \left( \int_c^t \frac{ds}{d\mu} |v|^p \right)^{1/p} \mu(t, \infty) < \infty.
\]

For a general tree \( \Gamma \), a necessary and sufficient condition for \( H_c \) to be bounded is given in [8].

**Remark 4.7** In [9] global estimates and an asymptotic formula were obtained for the approximation numbers of \( H_c \). Under appropriate conditions on \( u, v \), it is proved in particular, in [9, Theorem 6.4], that
\[
\lim_{\varepsilon \to 0} \varepsilon \nu(\varepsilon, H_c) = \lim_{m \to \infty} m a_m(H_c) = \alpha_p \int_{\Gamma} |uv| \, dt, \tag{4.37}
\]
where \( \alpha_p \) is given in (2.3); \( u \in L^{p'}(\Gamma), v \in L^p(\Gamma) \) are sufficient conditions for (4.37) to hold. Estimates for \( L^q \) and weak-\( L^q \) norms of the approximation numbers are also given in [9]. Analogous results for the case \( p = 2 \) are obtained in [10] by different techniques. The following result can be extracted from [9] to give significant information about the term involving \( \nu(c, \varepsilon, \Gamma, \varepsilon) \) on the right-hand side of (4.4). A comprehensive analysis of such results is given in [12], where, *inter alia*, a sharp upper bound for the approximation numbers of a Sobolev embedding is derived for metric graphs of finite length.

**Lemma 4.8** Let \( \Gamma \) be a tree with \( u \in L^{p'}(\Gamma) \) and \( v \in L^p(\Gamma) \) in (4.28). Then
\[
a_{N+4}(H_c) \leq 3\frac{\gamma_p}{N} \|u\|_{L^{p'}(\Gamma)} \|v\|_{L^p(\Gamma)}, \tag{4.38}
\]
where \( \gamma_p = 2 \) when \( p \neq 2 \) and \( \gamma_2 = 1 \).

**Proof.** A central role in the argument in [9] is played by two partitions of \( \Gamma \) into \( M(\Gamma, \varepsilon) \) and \( N(\varepsilon, \Gamma) \) subtrees \( \Gamma_i \) defined in terms of the quantity
\[
A(\Gamma_i) := \sup_{0 \neq f \in L^p(\Gamma)} \inf_{\alpha \in \mathbb{C}} \frac{\|H_c f - \alpha v\|_{L^p(\Gamma)}}{\|f\|_{L^p(\Gamma)}}.
\]
In [9, Lemmas 3.9 and 3.15] it is proved that
\[
N(\varepsilon, \Gamma) \leq 3[M(\Gamma, \varepsilon) + 1]
\]
and
\[
a_{N+1}(H_c) \leq \gamma_p \varepsilon, \quad N \equiv N(\varepsilon, \Gamma),
\]
where $\gamma_p = 2$ when $p \neq 2$ and $\gamma_2 = 1$. Hence,

$$\frac{\varepsilon}{3}(N - 3) \leq \varepsilon M(\Gamma, \varepsilon) < \sum_{i=1}^{M(\Gamma, \varepsilon)} A(\Gamma_i)$$

$$\leq \sum_{i=1}^{M(\Gamma, \varepsilon)} \|u\|_{p', \Gamma_i} \|v\|_{p, \Gamma_i}$$

and

$$\|u\|_{p', \Gamma} \|v\|_{p, \Gamma},$$

whence (4.38).

On substituting (4.38) in (4.4), and using Theorem 3.4 and Lemma 4.5, we get

**Corollary 4.9** Let the hypothesis of Theorem 4.1 hold and suppose that for each $\Omega_i(\varepsilon), i = 1, 2, \ldots, q(\varepsilon), u = (\frac{ds}{d\mu})^{1/p} \in L^p(\Gamma_i(\varepsilon))$. Then,

$$\varepsilon^n \{\nu_1(\varepsilon, \Omega) + 1\} \leq \delta^n \{\nu_1(\varepsilon, \Omega) + 1\} \Sigma_{j \in L_0}(Q_j(\varepsilon))$$

$$+ K\varepsilon^n \delta^{-n} \Sigma_{k=0}^{n-1} \Sigma_{j \in L_k}(p_j(\varepsilon, \delta, k))$$

$$+ K\varepsilon^{n-1} \Sigma_{i=1}^{q(\varepsilon)} \mu(\Gamma_i(\varepsilon))^{1/p} \|u\|_{p', \Gamma_i(\varepsilon)}$$

$$+ \varepsilon^n(q(\varepsilon) + h(\varepsilon)),$$

where $K$ is a positive constant.

5. A LOWER BOUND FOR $\mu_0(\varepsilon, \Omega)$

**Theorem 5.1** Let the hypothesis of Theorem 4.1 hold. Then,

$$\varepsilon^n \mu_0(\varepsilon, \Omega) \geq \delta^n \mu_0(\varepsilon, \Omega) \{\Sigma_{j \in L_0}(Q_j(\varepsilon)) - K\varepsilon^n \Sigma_{j \in L_0} \prod_{j=2}^{n} l_j(\varepsilon, j)\}$$

$$+ \Sigma_{i=1}^{q(\varepsilon)} \varepsilon^n \mu_0(\varepsilon, \Omega_i(\varepsilon)) + \Sigma_{i=q(\varepsilon)+1}^{q(\varepsilon)+h(\varepsilon)} \varepsilon^n \mu_0(\varepsilon, \Omega_i(\varepsilon)),$$  \hspace{1cm} (5.1)

where $\mu_0(\varepsilon, \Omega_i(\varepsilon)) \leq 1$ for $i = q(\varepsilon) + 1, \ldots, q(\varepsilon) + h(\varepsilon)$. Moreover,

$$\sup_{\delta > 0} \{\delta^n \mu_0(\delta, U)\} = \lim_{t \to 0^+} t^n \mu_0(t, U) =: L^0_U.$$ \hspace{1cm} (5.2)

Suppose that, for $i = 1, \ldots, q(\varepsilon)$

$$\lim_{\varepsilon \to 0} \varepsilon^{-p} \left\{\int_{\Gamma_i(\varepsilon)} \left(\int_{J_i} \phi^{-p'/p} ds\right)^{p'/p'} d\mu(t)\right\} = 0, \hspace{1cm} (5.3)$$
where \( J_t \) is a subtree of \( \Gamma_i(\varepsilon) \) containing \( t \) and \( \tau(B_t) \),

\[
\phi(s) := \int_{\{x: \tau(x) = s\}} |\nabla \tau(x)|^{p-1} d\theta(x)
\]

and \( d\theta(x) \) denotes \((n - 1)\)-dimensional Hausdorff measure. Then

\[
\mu_0(\varepsilon, \Omega_i(\varepsilon)) \geq \mu_0(\varepsilon \cdot \Gamma_i(\varepsilon)) \geq \frac{\alpha_p}{2c\varepsilon |\Gamma_i(\varepsilon)|}
\]

for some constant \( c \), where \( \alpha_p \) is given in (2.10).

**Proof.** It follows from [9, Lemma 5.4] that

\[
\varepsilon^n \mu_0(\varepsilon, \Omega) \geq \varepsilon^n \mu_0(\varepsilon, \Omega_0(\varepsilon)) + \varepsilon^n \sum_{i=1}^{q(\varepsilon)} \mu_0(\varepsilon, \Omega_i(\varepsilon)) + \frac{\varepsilon^n}{2c\varepsilon |\Gamma_i(\varepsilon)|} \geq \mu_0(\varepsilon, \Omega_i(\varepsilon))
\]

say. By an argument similar to that for (4.14) with respect to a rectangular block \( Q \), we obtain from [7, Lemma 5.4] that, for \( \varepsilon^{-1} \delta l \geq 1, \)

\[
\varepsilon^n \mu_0(\varepsilon, Q) \geq \varepsilon^n M_1 M_2 \cdots M_n \mu_0(\delta, U) = \varepsilon^n \Pi_{r=1}^{n} (\delta \varepsilon^{-1} l_r - \theta_r) \mu_0(\delta, U) = \mu_0(\delta, U) \left\{ \delta^n |Q| - O(\varepsilon \delta^{-1} \Pi_{s=2}^{n} l_r(\varepsilon, j)) \right\}.
\]

We therefore get

\[
J_1 \geq \sum_{j \in L_0} \varepsilon^n \mu_0(\varepsilon, Q_j(\varepsilon)) \geq \delta^n \mu_0(\delta, U) \left\{ \sum_{j \in L_0} |Q_j(\varepsilon)| - K \varepsilon \delta^{-1} \Pi_{s=2}^{n} l_s(\varepsilon, j) \right\}.
\]

For the terms in \( J_3 \), we obtain, as for those in \( I_3 \) in (4.6) and on using Lemma 5.1 in [7] that

\[
\mu_0(\varepsilon, \Omega_i(\varepsilon)) \leq \nu_1(\varepsilon, \Omega_i(\varepsilon)) + 1 \leq 1.
\]

Thus (5.1) is proved.

The inequality (5.4) is obtained by a similar argument to that used for (4.19) except that now we use instead of (4.19) the commuting diagram
the suffices "0" indicating that the compact support functions are dense in the spaces; in (5.9) \(\Omega'\) is a GRD and \(\Gamma'\) is an associated tree. We have

\[
M_0 E_0 = I_0 M_1 \\
E_0 T_1 = T_0 I_0.
\]  

Thus

\[
I_0 = I_0 - M_0 T_0 I_0 + M_0 T_0 I_0 = I_0 - M_0 T_0 I_0 + M_0 E_0 T_1.
\]  

For \(F \in L_0^{1,p}(\Gamma', d\mu)\),

\[
| (I_0 - M_0 T_0 I_0) F(t) | = | F(t) - \frac{1}{|B_t|} \int_{B_t} F(\tau(x)) dx | \\
= | \frac{1}{|B_t|} \int_{B_t} \{ \int_0^{\text{dist}(t,\tau(x))} \frac{d}{d\sigma} F(s(\sigma)) d\sigma \} dx |,
\]

where \(s = s(\sigma)\) is the natural parametrization of \(\Gamma'\) in terms of the polygonal distance \(\sigma\),

\[
\leq \int_0^{L_t} | \frac{d}{d\sigma} F(s(\sigma)) | d\sigma = \int_{J_t} | F'(s) | ds,
\]

where \(L_t = \max_{x \in B_t} \{ \text{dist}(t, \tau(x)) \}\) and \(J_t = \{ s = s(\sigma) : 0 < \sigma < L_t \}\). Then, in the notation (5.3) we have

\[
| (I_0 - M_0 T_0 I_0) F(t) | \leq \int_{J_t} | F'(s) | \phi(s)^{1/p} \phi(s)^{-1/p} ds \\
\leq \left( \int_{J_t} | F'(s) |^p \phi(s) ds \right)^{1/p} \left( \int_{J_t} \phi(s)^{-p'} ds \right)^{1/p'} \\
= \left( \int_{\tau^{-1}(J_t)} | \nabla T_1 F(x) |^p dx \right)^{1/p} \left( \int_{J_t} \phi(s)^{-p'} ds \right)^{1/p'}
\]
on using the co-area formula (see [10, Theorem 1.2.4]), namely
\[
\int_{\tau^{-1}(J_t)} |\nabla(T_1 F(x))|^p dx = \int_{\tau^{-1}(J_t)} |F'(\tau(x))|^p |\nabla \tau(x)|^p dx
\]
\[
= \int_{J_t} |F'(s)|^p \int_{\tau(x)=s} |\nabla \tau(x)|^{p-1} d\theta(x) ds
\]
Hence
\[
\| (I_0 - M_0 T_0 I_0) F \|_{L^p(\Gamma',d\mu)} \leq \left\{ \int_{\Gamma'} \left( \int_{J_t} \phi(s)^{-\frac{p}{p'}} ds \right)^{\frac{p}{p'}} d\mu(t) \right\}^{1/p} \| \nabla(T_1 F) \|_{p,\Omega'}
\]
\[
= c(\Gamma') \| \nabla(T_1 F) \|_{p,\Omega'}
\]
say. Suppose \( c(\Gamma') < \varepsilon/2\gamma \), where \( \gamma \) is the constant in Definition 1.1. Then, if \( S \) is a finite dimensional subspace of \( L^1_{0,p}(\Gamma',d\mu) \), and
\[
\alpha(S) := \sup S \left\{ \frac{\| F' \|_{L^p(\Gamma',d\mu)}}{\| I_0 F \|_{L^p(\Gamma',d\mu)}} \right\} \leq \frac{1}{\varepsilon}
\]
we have from (5.12) and [7, (3.6)]
\[
\| \nabla(T_1 F) \|_{p,\Omega'} \leq \gamma \| F' \|_{L^p(\Gamma',d\mu)} = \gamma \| F \|_{L^1_{0,p}(\Gamma',d\mu)}
\]
\[
\leq \gamma \varepsilon^{-1} \| I_0 F \|_{L^p(\Gamma',d\mu)}
\]
\[
\leq \gamma \varepsilon^{-1} \left\{ c(\Gamma') \| \nabla(T_1 F) \|_{p,\Omega'} + \| M_0 \| \| E_0 T_1 F \|_{p,\Omega'} \right\}
\]
and so, for some positive constant \( c \),
\[
\| E_0 T_1 F \|_{p,\Omega'} \geq c\varepsilon \| \nabla(T_1 F) \|_{p,\Omega'}.
\]
Consequently, since \( S \) and \( T_1 S \) have the same dimension,
\[
\mu_0(\varepsilon,\Omega') \geq \mu_0(\varepsilon,\Gamma').
\]
It follows from [7, Lemma 5.4] that, for any finite subset \( E_i \) of edges of \( \Gamma_i(\varepsilon) \),
\[
\mu_0(\varepsilon,\Gamma_i(\varepsilon)) \geq \sum_{e \in E_i} \mu_0(\varepsilon,e)
\]
\[
\geq \frac{\alpha_p}{2\varepsilon} \sum_{e \in E} |e|,
\]
by the first inequality in (2.11), whence (5.5). The identity (5.2) is proved in [7, Theorem 5.8]. The proof is therefore complete. \( \square \)

**Remark** When \( p = 2 \)
\[
L^0_U = L_U = (2\pi)^{-n} \omega_n,
\]
where \( L_U, L^0_U \) are the limits in (4.5) and (5.2) respectively, and \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \); see [7,(5.12)]. When \( n = 1 \), (2.12) yields
\[
L^0_U = \frac{\alpha_p}{2},
\]
c.f. Remark 4.3.
6. Example

We illustrate the applicability and efficacy of our results for a self-similar domain $\Omega$ in $\mathbb{R}^2$ which was analysed in [7]. It is constructed from a succession of finite sets (generations) $\Theta_m$ of closed congruent rectangles $Q_m$ of edge lengths $2\alpha_m \times 2\beta_m (\alpha_m \leq \beta_m)$ and with disjoint interiors; see Figure 1. The generation $\Theta_0$ consists of a single rectangle, as does $\Theta_1$, a short edge of $Q_1$ being attached to the middle portion of a long edge of $Q_0$. For $m \geq 1$, $\Theta_m$ contains $2^{m-1}$ rectangles and to each long edge of $Q_m \in \Theta_m$ is attached a short edge of a rectangle $Q_{m+1} \in \Theta_{m+1}$. The domain $\Omega$ is the interior of the set $\Theta$ constructed in this way:

$$\Omega = \Theta^0, \quad \Theta = \bigcup_{m \in \mathbb{N}_0} \left( \bigcup \{Q_m : Q_m \in \Theta_m\} \right).$$

Let

$$\alpha_m = c^{\alpha m}, \quad \beta_m = c^m, \quad m \in \mathbb{N}_0,$$

where $\alpha > 1, c^{\alpha} + 2c^2 < 1$ and, for $\alpha > p/p', c^{1+\alpha/p+p/\alpha^2} < 1/2$; these last two assumptions are to ensure that there is no overlapping. It is proved in [7, Section 6] that the embedding $E: W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. We shall consider the case when $2c > 1$. This case is of special interest as it is the most pathological. Its boundary has outer and inner Minkowski dimensions $d_o, d_i$ respectively, given by

$$d_o = \frac{\log 2}{\log(1/c)}, \quad d_i = 1 - \frac{1}{\alpha} + \frac{\log 2}{\alpha \log(1/c)}$$

(6.3)

and the upper Minkowski contents of $\Omega$ and its complement are finite; see [7, Section 6.2.2]. Furthermore, $1 < d_i < d_o < 2$.

Let

$$\Omega_m := \bigcup_{i \leq m} \{Q : Q \in \Theta_i\}, \quad \Omega \setminus \Omega_m = \bigcup_{i=1}^{2^m} A_{i,m}. \quad (6.4)$$

For each $i, A_{i,m}$ is a generalised ridged domain which is similar to $\Omega$. To apply Theorem 4.1, we choose $\Omega_0(\varepsilon) = \Omega_m$ for an appropriate choice of $m$, with the remainder of $\Omega$ made up of the GRDs $A_{i,m}$.

In $A_{i,m}$, the generalised ridge is the tree rooted at the point O in Figure 2, and the map $\tau$ is the projection of PA onto OL (or what corresponds to SL in the first rectangle) and BC onto LR. The following facts are established in [7, Section 6]. The symbol $\asymp$ means that the quotient of the two sides exhibited is bounded between positive constants.
• For \( x \in Q_m \) and \( t \) on the part of the generalised ridge in \( Q_m \)

\[
|\nabla \tau(x)| \asymp 1
\]

\[
\frac{d\mu}{dt} \asymp \alpha_m
\]

\[
|\rho'(t)| \leq 1.
\]

(6.5)

• \( \nu_1(\varepsilon, A_{i,m}) = 0 \) if \( m \geq m(\varepsilon) \) where

\[
m(\varepsilon) = \frac{\log(1/\varepsilon)}{\log(1/c)} + O(1).
\]

(6.6)

**Upper Bound**

We apply Theorem 4.1 with

\[
\Omega_0(\varepsilon) = \bigcup_{i \leq m(\varepsilon)} \{ Q : Q \in \Theta_i \}
\]

\[
\Omega_i(\varepsilon) = A_{i,m(\varepsilon)}, i = 1, \ldots q(\varepsilon).
\]

(6.7)

Note that the number \( n_i \) of rectangles in \( \Theta_i \) satisfies

\[
n_0 = 1, n_i = 2^{i-1}, i \geq 1 ,
\]

and hence in the notation of Theorem 4.1,

\[
r(\varepsilon) = 1 + \sum_{i=1}^{m(\varepsilon)} 2^{i-1}, \quad q(\varepsilon) = 2^{m(\varepsilon)}.
\]

From Theorem 4.1 we have

\[
\varepsilon^2 [\nu_1(\varepsilon, \Omega) + 1] \leq \varepsilon^2 \{\nu_1(\varepsilon, \Omega_0(\varepsilon)) + 1\}
\]

\[
\leq \delta^2 [\nu_1(\delta, U) + 1] |\sum_{j \in L_0} |Q_j(\varepsilon)|
\]

\[
+ K \varepsilon^2 \delta^{-2} |\sum_{k=0}^{l_1} \sum_{j \in L_k} p_j(\varepsilon, \delta, k) + \varepsilon^2 q(\varepsilon),
\]

(6.8)

where \( l_1(\varepsilon, j) = 2^d/\varepsilon^j \) and \( l_2(\varepsilon, j) = 2^d \). Furthermore, \( j \in L_0 \) if and only if \( l_1(\varepsilon, j) \geq \frac{c}{\alpha} \) and hence \( j \leq j_0 \) where

\[
j_0 = j_0(\varepsilon) := \frac{\log(2\delta/\varepsilon)}{\alpha \log(1/c)}.
\]

(6.9)

Thus from (6.8)

\[
\varepsilon^2 [\nu_1(\varepsilon, \Omega) + 1] \leq \delta^2 [\nu_1(\delta, U) + 1] |\sum_{j \leq j_0} n_j |Q_j|
\]

\[
+ K \varepsilon^2 \delta^{-2} \{\sum_{j \leq j_0} n_j p_j(\varepsilon, \delta, 0) + \sum_{j > j_0} n_j \}
\]

\[
+ \varepsilon^2 q(\varepsilon).
\]

(6.10)
In (6.10), on noting that for \( j \in L_0, \varepsilon \delta^{-1} < c^{\alpha j}/2 < 1 \), it readily follows that

\[
\begin{align*}
\Sigma_{j \leq j_0} n_j |Q_j| & \leq |\Omega_0(\varepsilon)| \\
\Sigma_{j \leq j_0} n_j p_j (\varepsilon, \delta, 0) & = \Sigma_{j=0}^{j_0} n_j \{ 4 \delta \varepsilon^{-1} (c^{\alpha j} + c^j) + 1 \} \\
& = O(\delta \varepsilon^{-1} n_{j_0}(2c)^j) \\
& = O\left( (\delta^{1+\frac{1}{p}(d_0-1)} \varepsilon^{-1} - \frac{1}{\alpha}(d_0-1)) \right) \\
& = O(\varepsilon^{-d_0}) \\
\Sigma_{j > j_0} n_j & \leq \Sigma_{j=1}^{m(\varepsilon)} q^{j-1} \\
& \leq 2^{m(\varepsilon)} = O(\varepsilon^{-d_0}).
\end{align*}
\]

Hence, we have

\[
\varepsilon^2 [\nu_1(\varepsilon, \Omega) + 1] \leq \delta^2 [\nu_1(\delta, U) + 1] |\Omega_0(\varepsilon)| + O(\delta^{-2} \varepsilon^{2-d_0}). \tag{6.12}
\]

In particular, this gives

\[
\limsup_{\varepsilon \to 0} \{ \varepsilon^2 \nu_1(\varepsilon, \Omega) \} \leq L_U |\Omega|. \tag{6.13}
\]

**Lower Bound**

We now take

\[
\Omega_0(\varepsilon) = \bigcup_{i \leq m-1} \{ Q : Q \in \Theta_i \}
\]

\[
\Omega_i(\varepsilon) = Q \in \Theta_m, \quad i = 1, 2, \ldots, q(\varepsilon), \tag{6.14}
\]

where \( m = m(\varepsilon) \). Thus the GRDs \( \Omega_i(\varepsilon) \) are the rectangles of side \( 2c^{am} \times 2c^{m} \) in \( \Theta_m \), and \( q(\varepsilon) = 2^m \). We therefore have, using \(| \cdot |\) to denote Lebesgue measure,

\[
|\Gamma_i(\varepsilon)| = 2(c^{am} + c^{m}).
\]

Also in (5.3), \(|J_t| \asymp \alpha_m = c^{am} \) and

\[
\phi(s) \asymp \int_{\{x : r(x) = s \in J_t\}} d\theta(x) \geq 2^{\alpha(m+1)}.
\]

Since \( d\mu/dt \asymp c^{am} \) from (6.5), it is readily seen that

\[
\varepsilon^{-p} \int_{\Gamma_i(\varepsilon)} \left( \int_{J_t} \phi^{-p/\nu} ds \right)^{p/\nu} \mu(t) = O(\varepsilon^{1+p \left( \frac{\alpha}{p} - 1 \right)}).
\]

Hence (5.3) is satisfied since \( 1 + p \left( \frac{\alpha}{p} - 1 \right) > 1 + p \left( \frac{1}{p} - 1 \right) = 0 \), and, in view of (5.5), we have for the GRDs \( \Omega_i(\varepsilon) \)

\[
\varepsilon^{2 \sum_{i=1}^{q(\varepsilon)} \mu_0(\varepsilon, \Omega_i(\varepsilon))} \geq K \varepsilon^{2m} (c^{am} + c^{m}) \geq K \varepsilon (2c)^m \asymp \varepsilon^{2-d_0}.
\]
Consequently Theorem 5.1 yields
\[
\varepsilon^2 \mu_0(\varepsilon, \Omega) \geq \delta^2 \mu_0(\delta, U) \left\{ \sum_{j \leq j_0} n_j |Q_j| - K \varepsilon \delta^{-1} \sum_{j \leq j_0} n_j l_2(\varepsilon, j) \right\} + K \varepsilon^{2-d_0}. \tag{6.15}
\]

In (6.15)
\[
\sum_{j \leq j_0} n_j |Q_j| = |\Omega| - O\left( \sum_{j > j_0} [2e^{1+\alpha}]^j \right)
= |\Omega| - O\left( \left( \frac{\varepsilon}{\delta} \right)^{1-\frac{d_0-1}{\alpha}} \right),
\]
and
\[
\varepsilon \delta^{-1} \sum_{j \leq j_0} n_j l_2(\varepsilon, j) = O\left( \varepsilon \delta^{-1} \sum_{j \leq j_0} (2e)^j \right)
= O\left( \left( \frac{\varepsilon}{\delta} \right)^{1-\frac{d_0-1}{\alpha}} \right).
\]
Since \( \delta^2 \mu_0(\delta, U) \) is bounded, for \( \delta \in (0, 1) \) by (5.2), Theorem 5.1 therefore gives
\[
\varepsilon^2 \mu_0(\varepsilon, \Omega) \geq \delta^2 \mu_0(\delta, U) |\Omega| - O\left( \left( \frac{\varepsilon}{\delta} \right)^{1-\frac{d_0-1}{\alpha}} \right) + K \varepsilon^{2-d_0}. \tag{6.16}
\]
Note that \( 1 - \frac{d_0-1}{\alpha} > 2 - d_0 > 0 \). On choosing \( \delta = \delta(\varepsilon) \), where
\[
\delta(\varepsilon)^{1-\frac{d_0-1}{\alpha}} = \varepsilon^{(1-\frac{1}{\alpha})(d_0-1)-\theta}
\]
for \( 0 < \theta < (1 - \frac{1}{\alpha})(d_0-1) \), we have
\[
\left( \frac{\varepsilon}{\delta} \right)^{1-\frac{d_0-1}{\alpha}} = \varepsilon^{2-d_0+\theta}.
\]
Hence, we have as \( \varepsilon \to 0 \),
\[
\varepsilon^2 \mu_0(\varepsilon, \Omega) \geq \delta(\varepsilon)^2 \mu_0(\delta(\varepsilon), U) |\Omega| + K \varepsilon^{2-d_0} - O\left( \varepsilon^{2-d_0+\theta} \right), \tag{6.17}
\]
where \( K \) and \( \theta \) are positive constants. In [7, Theorem 6.3] it is proved that when \( p = 2 \),
\[
\varepsilon^2 \mu_0(\varepsilon, \Omega) - \varepsilon^{2-d_0} \geq \frac{1}{4\pi} |\Omega| \gg \varepsilon^{2-d_0}. \tag{6.18}
\]

**References**


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