NEUMANN LAPLACIANS ON DOMAINS AND OPERATORS ON ASSOCIATED TREES

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Abstract

Connections are established between the essential spectra of the Neumann Laplacian on a domain \( \Omega \) in \( \mathbb{R}^2 \) and a one-term operator of Sturm–Liouville type which is defined naturally on the skeleton, or a generalized ridge, \( \Gamma \) of \( \Omega \), when \( \Gamma \) is a tree. Horns, spirals, 'rooms and passages' and domains with fractal boundaries, like the Koch snowflake, are examples of such domains \( \Omega \).

The analysis hinges on the existence of isometric maps between \( L^2(\Omega) \), \( H^1(\Omega) \) and weighted \( L^2 \), \( H^1 \) spaces defined on \( \Gamma \) in terms of a Lipschitz map \( \tau \) which projects \( \Omega \) onto \( \Gamma \).

1. Introduction

In [4, 5, 6] properties of the Sobolev embedding \( E : W^{1,p}(\Omega) \to L^p(\Omega), \ 1 < p < \infty \), were studied for a wide class of domains \( \Omega \) in \( \mathbb{R}^n \) which includes ones with irregular boundaries like horns (referred as trumpets by Evans and Harris [4]), spirals, rooms and passages and snowflakes. Specifically, two-sided estimates were derived for the quantity

\[
\beta(E) := \inf \{ \| E - P \| : P \in \mathcal{K} \},
\]

where \( \mathcal{K} \) denotes the set of compact operators from \( W^{1,p}(\Omega) \) into \( L^p(\Omega) \), and these gave, in particular, necessary and sufficient conditions for \( E \) to be compact, i.e. \( \beta(E) = 0 \). The case \( p = 2 \) is of special interest as the Neumann Laplacian \( -\Delta \) is defined, in the weak sense, by \( (1 - \Delta)^{-1} = EE^* \), where \( E^* : L^2(\Omega) \to W^{1,2}(\Omega) \) is the adjoint of \( E \), so that \( -\Delta \) has a discrete spectrum if and only if \( E \) is compact. The technique used in the aforementioned papers rested on the established equivalence of the validity of the Poincaré inequality for \( W^{1,p}(\Omega) \) and that of an analogous inequality holding on the image in \( \Omega \) (called the generalized ridge) of an interval or tree \( \Gamma \) under a Lipschitz continuous map \( u \); the existence of a generalized ridge is a characteristic feature of the class of domains considered. A typical example of a generalized ridge is the skeleton of \( \Omega \) when it is the Lipschitz image of a tree (see Section 2 for details).

The present paper continues the investigation in [4, 5, 6] in that the objective is to examine how the essential spectrum \( \sigma_e(-\Delta) \) of \( -\Delta \) depends on what happens on \( u(\Gamma) \). The main results make comparisons between \( \sigma_e(-\Delta) \) and the essential spectrum \( \sigma_e(H_\Gamma) \) of an operator \( H_\Gamma \) of one-term Sturm–Liouville type which is defined naturally on \( \Gamma \). In some instances we can prove that the two

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essential spectra coincide: this is the case in \cite{1} where \( \Omega \) is a horn and it is shown that, in appropriate circumstances, there is absolutely continuous spectrum; see also \cite{9}. The relationships between \(-\Delta_{\Omega}\) and \(H_{\Gamma}\) when \( \Omega \) is a horn for which the spectrum of \(-\Delta_{\Omega}\) is discrete is also featured in \cite{10} where the asymptotic distribution of the eigenvalues of \(-\Delta_{\Omega}\) is investigated; see also \cite{2}.

In general, however, the situation is somewhat delicate. The inclusion \( \sigma_e(-\Delta_{\Omega}) \subseteq \sigma_e(H_{\Gamma}) \) is proved under an assumption which is mild as long as \( \Omega \) is unbounded and its boundary does not have features like inward-pointing corners. But the reverse inclusion \( \sigma_e(-\Delta_{\Omega}) \subseteq \sigma_e(H_{\Gamma}) \) seems difficult to confirm or deny; we can only prove, in general, that \( \sigma_e(H_{\Gamma}) \) lies in a set which is concentric to \( \sigma_e(-\Delta_{\Omega}) \) (see Theorem 5.12), the dilation depending on the ultimate behaviour of operators at the vertices of \( \Gamma \).

2. Preliminaries

Let \( \Gamma \) be a tree in \( \mathbb{R}^2 \), that is, a connected graph in \( \mathbb{R}^2 \) without loops or cycles, where its edges are non-degenerate open line segments whose end points are the vertices. We shall assume that \( \Gamma \) has, at most, a countably infinite number of edges, and each vertex of \( \Gamma \) is of finite degree, that is, only a finite number of edges emanate from each vertex. For every \( x, y \in \Gamma \) there is a unique polygonal path in \( \Gamma \) joining \( x \) and \( y \). The distance between \( x \) and \( y \) is defined to be the length of the polygonal path; in this way \( \Gamma \) is endowed with a metric topology \( \pi(\Gamma) \). It is shown in \cite[Lemma 2.1]{6} that if \( \Gamma \) contains an infinite number of vertices, \( \pi(\Gamma) \) is locally compact, a closed set \( A \subset \Gamma \) being compact if and only if it meets only a finite number of edges. Also, if \( \Gamma \) is endowed with the natural one-dimensional Lebesgue measure, it is a \( \sigma \)-finite measure space.

For \( a \in \Gamma \) we define \( t \geq_a x \) (or equivalently \( x \leq_a t \)) to mean that \( x \) lies on the path from \( a \) to \( t \); we write \( x <_a t \) for \( x \leq_a t \) and \( x \neq t \). This is a partial ordering on \( \Gamma \) and the ordered graph so formed is referred to as a tree rooted at \( a \). If \( a \) is not a vertex we make it one by replacing the edge on which it lies by two edges. In this way \( \Gamma \) is the unique finite union of subtrees which intersect only at \( a \).

The path joining two points \( x, y \in \Gamma \) may be parametrized by \( \sigma(t) = \text{dist}(x, t) \) and for \( g \in L^1_{\text{loc}}(\Gamma) \) we have

\[
\int_x^y g(t) \, dt = \int_0^{\text{dist}(x, y)} g(t(\sigma)) \, d\sigma,
\]

where \( t(\sigma) \in \Gamma \) is defined by \( \text{dist}(x, t(\sigma)) = \sigma \).

Let \( \Omega \) be a domain (i.e. a connected open set) in \( \mathbb{R}^2 \) with the following properties.

ASSUMPTION 2.1. (i) There exists a tree \( \Gamma \) and a function \( \tau \) mapping \( \overline{\Omega} \) onto \( \Gamma \) which is locally Lipschitz, i.e., for each \( x \in \Omega \) there exists a neighbourhood \( V(x) \subset \Omega \) of \( x \) and a positive constant \( \gamma(x) \) such that for all \( y \in V(x) \)

\[
d\tau(\tau(x), \tau(y)) \leq \gamma(x)|x - y|,
\]

where \( d\tau \) denotes the metric on \( \Gamma \) and \( |\cdot| \) the Euclidean metric (for definiteness) on \( \mathbb{R}^2 \); we write \( \tau \in \text{Lip}_{\text{loc}}(\overline{\Omega}, \Gamma) \).

(ii) \( C(t) = \{ x \in \Omega : \tau(x) = t \} \) is a rectifiable curve for \( t \in \Gamma \) and there exists \( u : \Gamma \rightarrow \Omega \) such that \( u \in \text{Lip}_{\text{loc}}(\Gamma, \Omega) \), \( C(t) \cap u(\Gamma) = u(t) \) and \( C(t) \setminus u(\Gamma) \) has two components, \( C_\pm(t) \) say.

(iii) Let \( C_{\pm}(t) \) be parametrized by arc length \( s \) which is measured from \( u(t) \) with \( 0 \leq s \leq \ell_+(t) \) on \( C_+(t) \) and \(-\ell_-(t) \leq s \leq 0 \) on \( C_-(t) \); suppose that \( \ell_\pm \) are Lipschitz continuous and positive on the closure of each edge of \( \Gamma \).
(iv) Let \( \Omega' = \tau^{-1}(V(\Gamma)) \), where \( V(\Gamma) \) is the set of vertices of \( \Gamma \) and \( \Omega'' = \Omega' \cup u(\Gamma) \). Suppose that \( \tau \in C^1(\Omega, \Omega'') \); note that, as the countable union of rectifiable curves, \( \Omega'' \) has zero (two-dimensional) measure.

(v) Let \( \tau(x) = (\tau_1(x), \tau_2(x)) \in \Gamma \) and \(|\nabla \tau(x)| = |(\nabla \tau_1(x))^2 + (\nabla \tau_2(x))^2|^{1/2} \). Suppose that \(|\nabla \tau| \), \(|\nabla \tau|^{-1} \in L^1(C(t)) \) for all \( t \in \Gamma \).

The generalized ridged domains defined in [4] and [6] satisfy the above properties, as well as others which ensure that the domain \( \Omega \) is such that the Sobolev space \( W^{1,p}(\Omega) \), \( 1 < p < \infty \), is compactly embedded in \( L^p(\Omega_0) \) for some appropriate \( \Omega_0 \subset \Omega \). The properties (2.1) bring to mind those proved in [7] for the skeleton of \( \Omega \), namely the set

\[
\{ x \in \Omega : \exists y_1, y_2 \in \partial \Omega \text{ such that } y_1 \neq y_2 \text{ and } |x - y_1| = |x - y_2| = \text{dist}(x, \mathbb{R}^2 \setminus \Omega) \}.
\]

Amongst the many interesting results established in [7], it is proved that, if \( \Omega \) does not include a half-plane, then its skeleton is connected [7, Theorem 1B], and is expressible as the union of countably many paths of finite length [7, Proposition 2K]. If the skeleton is the Lip_{loc}-image of a tree, \( u(\Gamma) \) in our notation, and there are precisely two near points \( y_1, y_2 \) in the definition, then the properties (2.1) are satisfied with \( \ell_+ (t) = \ell_- (t) = \text{dist}(u(t), \mathbb{R}^2 \setminus \Omega) \). Note that the ridge of \( \Omega \) defined in [4], which motivated the notion of generalized ridged domain, is the central set of \( \Omega \) discussed in [7], and contains the skeleton.

Let \( E(\Gamma) = \{ e_j : j \in J \} \) denote the set of all edges \( e_j \) of \( \Gamma \). If we write \( e_j = (a_j, b_j) \), it is to be understood that \( a_j \leq_\sigma b_j \). Denoting distance along an edge by \( \sigma \), we adopt the notation, for any measurable subset \( \Gamma_0 \) of \( \Gamma \) and \( F \in L^1(\Gamma_0) \),

\[
\int_{\Gamma_0} F(t) \, dt = \sum_{j \in J} \int_{e_j} F(\sigma) \chi_{\Omega_0}(\sigma) \, d\sigma,
\]

where \( \chi_{\Omega_0} \) is the characteristic function of \( \Gamma_0 \); thus for simplicity we have written \( F(\sigma) \) in (2.2) rather than \( F(t(\sigma)) \).

If \( x = (x_1, x_2) \in \Omega \) is such that \( \tau(x) = t(\sigma) \in e \in E(\Gamma) \), then a coordinate system on \( \tau^{-1}(e) \), and hence on \( \Omega \), is defined by

\[
x = x(\sigma, s), \quad \tau(x) = t(\sigma), \quad s \in (-\ell_-(\sigma), \ell_+(\sigma)),
\]

where \( \ell_\pm (\sigma) = \ell_\pm (t(\sigma)) \). It readily follows that

\[
\frac{\partial(x_1, x_2)}{\partial(\sigma, s)} = \frac{1}{|\nabla \tau(\sigma, s)|},
\]

where \(|\nabla \tau(\sigma, s)| \) is given in (2.1) (v) with \( x = x(\sigma, s) \). For, with \( \tau(x) = (\tau_1(x), \tau_2(x)) \) and \( t(\sigma) = (t_1(\sigma), t_2(\sigma)) \), we have from (2.3)

\[
\frac{dt_k}{d\sigma} = \left\{ \begin{array}{ll}
\sum_{j=1}^{2} \frac{\partial \tau_k(x)}{\partial x_j} \frac{\partial x_j}{\partial \sigma}, \\
0 = \sum_{j=1}^{2} \frac{\partial \tau_k(x)}{\partial x_j} \frac{\partial x_j}{\partial s}
\end{array} \right.
\]

(2.5)
for $k = 1, 2$. Moreover, since $\sigma$, $s$ represent the arc lengths of $e$ and $C(t)$, respectively,
\[
\begin{align*}
\left| \frac{dt}{d\sigma} \right|^2 &= \left( \frac{dt_1}{d\sigma} \right)^2 + \left( \frac{dt_2}{d\sigma} \right)^2 = 1, \\
\left| \frac{dx}{ds} \right|^2 &= \left( \frac{dx_1}{ds} \right)^2 + \left( \frac{dx_2}{ds} \right)^2 = 1.
\end{align*}
\]
(2.6)

Equation (2.4) follows from (2.5) and (2.6).

For a measurable subset $\Gamma_0$ of $\Gamma$, let $\Omega_0 = \tau^{-1}(\Gamma_0)$ and $f \in L^1(\Omega_0)$ with $f = 0$ outside $\Omega_0$.

It then follows from above that
\[
\int_{\Omega_0} f(x) dx = \sum_{j \in J} \int_e \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} f(\sigma, s) \frac{1}{|\nabla \tau(\sigma, s)|} ds;
\]
(2.7)

note that we have again simplified the notation by writing $(\sigma, s)$ for $x(\sigma, s)$. This is also a consequence of the co-area formula
\[
\int_{\tau^{-1}(e)} f(x) dx = \int_e \int_{\tau^{-1}(\tau(\sigma))} \frac{f(x)}{|\nabla \tau(x)|} d\mathcal{H}^1(x),
\]
(2.8)

where $\mathcal{H}^1$ denotes one-dimensional Hausdorff measure, and the Assumptions 2.1(ii) and (iii).

Of particular importance is the case when $f = F \circ \tau$ in (2.7) with $F \in L^1(\Gamma_0)$ which is supported in $\Gamma_0$:
\[
\int_{\Omega_0} F \circ \tau(x) dx = \sum_{j \in J} \int_e \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} \frac{F(\sigma)}{|\nabla \tau(\sigma, s)|} ds
\]
\[
\quad =: \int_{\Gamma_0} F(\sigma) \alpha(\sigma) d\sigma,
\]
(2.9)

where
\[
\alpha(\sigma) := \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \frac{1}{|\nabla \tau(\sigma, s)|} ds, \quad t(\sigma) \in E(\Gamma).
\]
(2.10)

**Example 2.2** (Horn-shaped domain, Fig. 1). Let $\Omega$, $\Gamma$ and $u(\Gamma)$ be as follows:
\[
\begin{align*}
\Omega &= \{ x = (x_1, x_2) : x_1 > 0, -\ell-(x_1) < x_2 < \ell+(x_2) \}, \\
\Gamma &= [0, \infty), \\
u(\Gamma) &= \{ t = (t_1, 0) : t_1 > 0 \}.
\end{align*}
\]

Here, $\Gamma$ consists of a single edge and vertices at 0 and “infinity”; strictly speaking $\Gamma$ is not a tree since trees have edges of finite length, but this has no effect on the analysis in the paper. Alternatively, we can write $\Gamma = \bigcup_{j \in J} e_j$, $e_j = (j-1, j)$. The map $\tau$ is given by $\tau(x_1, x_2) = x_1$, and hence $|\nabla \tau(x)| = 1$ for all $x \in \Omega$.

**Example 2.3** (Rooms and passages, Fig. 2). Let $\{h_k\}$, $\{\delta_k\}$, $k = 1, 2, \ldots$, be infinite sequences of positive numbers such that
\[
\sum_{k=1}^{\infty} h_k = b < \infty, \quad 0 < \text{const.} \leq \frac{h_{k+1}}{h_k} \leq 1, \quad 0 < \delta_{2k} \leq h_{2k+1}.
\]
and let $H_k := \sum_{j=1}^{k} h_j$, $k = 1, 2, \ldots$. Then $\Omega \subset \mathbb{R}^2$ is defined as the union of the rooms $R_k$ and passages $P_{k+1}$ given by
\[
\begin{align*}
R_k &= (H_k - h_k, H_k) \times \left(-\frac{h_k}{2}, \frac{h_k}{2}\right), \\
P_{k+1} &= [H_k, H_k + h_{k+1}] \times \left(-\frac{\delta_{k+1}}{2}, \frac{\delta_{k+1}}{2}\right),
\end{align*}
\]
for $k = 1, 3, 5, \ldots$. In [4], this was analysed as an example of a generalized ridged domain, with generalized ridge $\Gamma = [0, b]$, $u(t) = (t, 0)$, and mapping $\tau$ defined as follows:
(i) in a passage $P$: $\tau(x_1, x_2) = x_1$;
(ii) in the first half of the room $R$ succeeding the passage $P$:
\[
\tau(x_1, x_2) = \max \left(x_1, |x_2| - \frac{\delta}{2}\right), \quad 0 \leq x_1 \leq \frac{b}{2},
\]
where $P$ is of width $\delta$ and $0 \leq x_1 \leq b$ in $R$ after translation. Hence $\tau$ is Lipschitz and $|\nabla \tau| = 1$.

The above choice of $\Gamma$ proved to be very effective in [4] for establishing necessary and sufficient criteria for the compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. In Section 6 we shall
examine other choices of \( \tau \) and \( \Gamma \). Of particular interest is the case when \( u(\Gamma) \) is the skeleton of \( \Omega \). Within a passage \( P \) and succeeding room \( R \), the skeleton \( u(\Gamma) \) is as shown in Fig. 3. Each point on \( u(\Gamma) \) is equidistant to precisely two points on \( \partial \Omega \), and the two components \( C_\pm(t) \) of \( \tau^{-1}(t) \) are the straight lines joining \( u(t) \) and the near points on \( \partial \Omega \). Thus \( u(\Gamma) \) consists of straight line segments and parabolic arcs like \( AC \) and \( CB \).

However, in this case \( \nabla \tau \) has a non-integrable singularity at the re-entrant corners of \( \Omega \). The re-entrant corners have to be rounded off to allow \( \Omega \) to be covered by our theory; one would not expect this to have much effect on the essential spectrum of \( -\Delta_\Omega \), but this needs proper investigation.

3. The spaces and operators

We denote by \( H^1(\Omega) \) the Hilbert space of complex-valued functions \( f \in L^2(\Omega) \) with weak first derivatives in \( L^2(\Omega) \) and having the inner product and norm

\[
(f, g)_{1,\Omega} := \int_\Omega (\nabla f \cdot \nabla g + f \overline{g}) \, dx,
\]

\[
\|f\|_{1,\Omega} := (f, f)_{1/2,\Omega}^{1/2}.
\]

We shall denote the usual \( L^2(\Omega) \) inner product and norm by \( (f, g)_\Omega \) and \( \|f\|_\Omega \), respectively. The natural embedding \( E : H^1(\Omega) \hookrightarrow L^2(\Omega) \) is injective, has dense range and \( \|E\| \leq 1 \).

The Neumann Laplacian \( -\Delta_\Omega \) on \( \Omega \) is defined by

\[
([ -\Delta_\Omega + 1] f, E\phi)_{\Omega} = (g, \phi)_{1,\Omega} \quad (\phi \in H^1(\Omega))
\]

for

\[ f = Eg \in \mathcal{D}(-\Delta_\Omega) = \{f : f \in \mathcal{R}(E), f, -\Delta_\Omega f \in L^2(\Omega)\}, \]

where \( \mathcal{D}(S) \) and \( \mathcal{R}(S) \) denote the domain and range of \( S \), respectively; that is, for \( f = Eg, g \in H^1(\Omega) \), the Neumann Laplacian \( -\Delta_\Omega f \) satisfies (3.2) for all \( \phi \in H^1(\Omega) \). In (3.2) we have distinguished between \( H^1(\Omega) \) and its image under \( E \) in \( L^2(\Omega) \). This might appear excessively pedantic, but we shall reap our reward later.

On writing \( h = (-\Delta_\Omega + 1) f \) in (3.2), we have, for all \( \phi \in H^1(\Omega) \),

\[
(g, \phi)_{1,\Omega} = (E^*h, \phi)_{1,\Omega}.
\]
where \( f = E g \) and \( E^* : L^2(\Omega) \to H^1(\Omega) \) is the adjoint of \( E \), and hence \( g = E^* h \) or \( E^{-1}(-\Delta_\Omega + 1)^{-1} h = E^* h \). It follows that
\[
(-\Delta_\Omega + 1)^{-1} = EE^* = |E^*|^2, \tag{3.3}
\]
where \(|E^*|\) is the absolute value of \( E^* \). The positive square root satisfies
\[
(-\Delta_\Omega + 1)^{-1/2} = |E^*| : L^2(\Omega) \to R(|E^*|) = R(E), \tag{3.4}
\]
and hence
\[
E^{-1}(-\Delta_\Omega + 1)^{-1/2} : L^2(\Omega) \to H^1(\Omega). \tag{3.5}
\]
In fact, (3.5) is an isometry onto \( H^1(\Omega) \). For if \( U \) is the partial isometry in the polar decomposition of \( E \), we have \(|E^*| = EU^*\), and
\[
|E^*|(E^{-1})^*E^{-1}|E^*| f = UU^* f = f
\]
for \( f \in R(E) = L^2(\Omega) \); see [3, Section IV.3]. Thus
\[
\|E^{-1}(-\Delta_\Omega + 1)^{-1/2} f\|_{1, \Omega}^2 = \|E^{-1}|E^*| f\|_{1, \Omega}^2 = (f, |E^*|(E^{-1})^*E^{-1}|E^*| f)_\Omega = \|f\|_{1, \Omega}^2. \tag{3.6}
\]
On \( \Gamma \), the underlying Hilbert space is \( L^2(\Gamma) \) which consists of (equivalent classes of) functions \( F \) which are measurable on \( \Gamma \) and satisfy (see (2.9))
\[
\int_\Gamma |F(\sigma)|^2 \alpha(\sigma) d\sigma := \sum_{j \in J} \int \ell_j^+ \nu_j^{-1/2} \alpha(\sigma) d\sigma < \infty; \tag{3.7}
\]
the associated inner-product and norm are
\[
\begin{align*}
(F, G)_\Gamma := & \int_\Gamma F(\sigma)\overline{G(\sigma)}\alpha(\sigma) d\sigma, \\
\|f\|_\Gamma := & \left( (F, F)_\Gamma \right)^{1/2}.
\end{align*} \tag{3.8}
\]
From (2.9), it follows that the map
\[
T_0 : F \to F \circ \tau : L^2(\Gamma) \to L^2(\Omega) \tag{3.9}
\]
is an isometry:
\[
\|T_0 F\|_\Omega = \|F\|_\Omega. \tag{3.10}
\]
Its adjoint \( T_0^* : L^2(\Omega) \to L^2(\Gamma) \) therefore has unit norm.

**Lemma 3.1.** The operator \( T_0^* \) is given by
\[
(T_0^* g)(\sigma) = \frac{1}{\alpha(\sigma)} \int_{t_+}^{t_-(\sigma)} g(\sigma, s) \frac{1}{|\nabla \tau(\sigma, s)|} ds, \quad g \in L^2(\Omega) \tag{3.11}
\]
for \( t(\sigma) \in e \in E(\Gamma) \).
Proof. For \( F \in L^2(\Gamma) \), (2.7) yields
\[
(T^* \mathbb{g}, F)_\Gamma = (g, T\mathbb{g})_\Omega = \sum_{j \in J} \int_{e_j} d\sigma \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} g(\sigma, s) \bar{F}(\sigma) \frac{1}{|\nabla \tau(\sigma, s)|} ds = \sum_{j \in J} \int_{e_j} \bar{F}(\sigma) \left\{ \frac{1}{\alpha(\sigma)} \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} g(\sigma, s) \frac{1}{|\nabla \tau(\sigma, s)|} ds \right\} \alpha(\sigma) d\sigma,
\]
and hence (3.11).

We denote by \( H^1(\Gamma) \equiv H^1(\Gamma; \alpha, \beta) \) the Hilbert space of functions \( F \in L^2(\Gamma) \) which satisfy the following conditions.

(3.12)(i) For each \( e \in E(\Gamma) \), \( F \in H^1(e; \alpha, \beta) \), the weighted Sobolev space on \( e \) with norm
\[
\left[ \int_e \left\{ \left| \frac{dF}{d\sigma} \right|^2 \beta(\sigma) + \left| F(\sigma) \right|^2 \alpha(\sigma) \right\} d\sigma \right]^{1/2},
\]
where
\[
\beta(\sigma) := \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla \tau(\sigma, s)| ds,
\]
and
\[
\frac{dF}{d\sigma} = \nabla F \frac{dt}{d\sigma} \quad (t(\sigma) \in e)
\]
is the directional derivative along \( e \) and
\[
\left| \frac{dF}{d\sigma} \right| = |\nabla F|;
\]
(3.12)(ii) \( F \) is continuous on \( \Gamma \);
(3.12)(iii)
\[
\sum_{j \in J} \int_{e_j} \left| \frac{dF}{d\sigma} \right|^2 \beta(\sigma) d\sigma < \infty.
\]

The inner product on \( H^1(\Gamma) \) is
\[
(F, G)_{1, \Gamma} := \sum_{j \in J} \left\{ \int_{e_j} \frac{dF}{d\sigma} \frac{dG}{d\sigma} \beta(\sigma) d\sigma + \int_{e_j} F(\sigma) \bar{G}(\sigma) \alpha(\sigma) d\sigma \right\} = \int_{\Gamma} \frac{dF}{d\sigma} \frac{dG}{d\sigma} \beta(\sigma) d\sigma + (F, G)_\Gamma, \quad (3.13)
\]
and we set \( \|F\|_{1, \Gamma} := (F, F)_{1, \Gamma}^{1/2} \).

The map (3.9) also defines an isometry between \( H^1(\Gamma) \) and \( H^1(\Omega) \). Specifically we have the following lemma.
Lemma 3.2. For $F \in H^1(\Gamma)$

$$\int_{\Omega} |\nabla(F \circ \tau)(x)|^2 \, dx = \sum_{j \in J} \int_{e_j} |F'(\sigma)|^2 \beta(\sigma) \, d\sigma. \quad (3.14)$$

Proof. On applying the chain rule to $f(x_1, x_2) = F(t_1, t_2)$, $t_k = \tau_k(x_1, x_2)$, $k = 1, 2$, we obtain

$$\frac{\partial f}{\partial x_k} = \nabla F \cdot \frac{\partial \tau}{\partial x_k},$$

and hence

$$|\nabla f| = |\nabla F| |\nabla \tau|.$$

For $t = t(\sigma) \in e \in E(\Gamma)$, we have

$$\frac{dF(\sigma)}{d\sigma} = \nabla F \cdot \frac{dt(\sigma)}{d\sigma}.$$

Since $|dt/d\sigma| = 1$, it follows that

$$|\nabla(F \circ \tau)(x)| = \left| \frac{dF}{d\sigma} \right| |\nabla \tau(\sigma, s)|, \quad x = x(\sigma, s),$$

and therefore, by (2.7),

$$\int_{\tau^{-1}(e)} |\nabla(F \circ \tau)(x)|^2 \, dx = \int_e \left| \frac{dF}{d\sigma} \right|^2 \, d\sigma \int_{\ell^+}(\sigma) |\nabla \tau(\sigma, s)| \, ds,$$

whence (3.14).

It follows from (3.10) and Lemma 3.2 that the map

$$T_1 : F \mapsto F \circ \tau : H^1(\Gamma) \to H^1(\Omega) \quad (3.15)$$

is an isometry:

$$\|T_1 F\|_{1, \Omega} = \|F\|_{1, \Gamma}, \quad (F \in H^1(\Gamma)). \quad (3.16)$$

Its adjoint $T_1^* : H^1(\Omega) \to H^1(\Gamma)$ has unit norm.

We denote by $I$ the natural embedding $I : H^1(\Gamma) \hookrightarrow L^2(\Gamma)$. It is an injection, has dense range and $\|I\| \leq 1$. The diagram in Fig. 4 is a reminder of the domain and target spaces of the maps introduced so far. We shall denote the identity on any of the function spaces by 1, the space being mentioned only if there is a likelihood of confusion.

In view of (3.10) and (3.16) it follows that

$$T_0^* T_0 = 1 \text{ on } L^2(\Gamma), \quad (3.17)$$

$$T_1^* T_1 = 1 \text{ on } H^1(\Gamma). \quad (3.18)$$

We also have in Fig. 4

$$ET_1 = T_0 I. \quad (3.19)$$
Furthermore, the operators

$$P_0 = T_0 T_0^*, \quad P_1 = T_1 T_1^*$$

are orthogonal projections on $L^2(\Omega)$ and $H^1(\Omega)$, respectively.

The non-negative self-adjoint operator in $L^2(\Gamma)$ associated with $(\cdot, \cdot)_{1,\Gamma}$ will be denoted by $H_\Gamma$; it is uniquely defined by (cf. (3.2))

$$([H_\Gamma + 1]F, I\Psi)_{1,\Gamma} = (G, \Psi)_{1,\Gamma} \quad (\Psi \in H^1(\Gamma))$$

for $F = IG \in \mathcal{D}(H_\Gamma) = \{F : F \in \mathcal{R}(I), \ F, H_\Gamma F \in L^2(\Gamma)\}$. It follows as in (3.3) that

$$(H_\Gamma + 1)^{-1} = II^* = |I^*|^2.$$  

**Theorem 3.3.** Let $F$ be a complex-valued function on $\Gamma$. Then $F \in \mathcal{D}(H_\Gamma)$ if and only if $F$ satisfies the following properties:

(a) $F \in \mathcal{R}(I)$,

(b) Let $F = IG$ with $G \in H^1(\Gamma)$; then $\beta G'$ is differentiable, in the weak sense, on each edge $e \in E(\Gamma)$ and

$$\sum_{j \in J_0} \int_{e_j} \left| \frac{d}{d\sigma} \left( \beta(\sigma) \frac{dG}{d\sigma} \right) \right|^2 \frac{d\sigma}{\alpha(\sigma)} < \infty.$$  

(c) We have

$$\langle H_\Gamma F(\sigma) \rangle = -\frac{1}{\alpha(\sigma)} \frac{d}{d\sigma} \left( \beta(\sigma) \frac{dG}{d\sigma} \right), \quad F = IG, t(\sigma) \in e \in E(\Gamma).$$

(d) Let $c$ be a vertex of $\Gamma$, $c = \cap_{j \in J_0} e_j$ say; then

$$\lim_{\sigma \to 0} \sum_{j \in J_0} \epsilon(\sigma) \beta(\sigma) \frac{dG}{d\sigma}(\sigma) = 0,$$

where $\epsilon(\sigma) = 1$ if $c \leq a t(\sigma)$ for all $t(\sigma) \in e_j$ and $-1$ otherwise.
Proof. (I) Suppose that $F \in \mathcal{D}(H_{\Gamma})$ such that $F = IG$ with $G \in H^1(\Gamma)$. (a) follows from the definition of $H_{\Gamma}$. Let $\Psi \in C_0^\infty(\Gamma) \cap H^1(\Gamma)$ with support in $e$. Then, from (3.21),

$$(H_{\Gamma} F, I \Psi)_\Gamma = \int_e \frac{dG}{d\sigma} \cdot \frac{d\Psi}{d\sigma} \beta(\sigma) d\sigma,$$

whence

$$\frac{d}{d\sigma} \left( \frac{dG}{d\sigma} \beta(\sigma) \right) = -\alpha(\sigma)(H_{\Gamma} F)(\sigma),$$

where the left-hand side is the weak derivative. The remainder of (b) follows since $H_{\Gamma} F \in L^2(\Gamma)$. Also we have proved that $H_{\Gamma} F$ satisfies (3.23). Let $\Psi \in C_0^\infty(\Gamma) \cap H^1(\Gamma)$ be supported in a neighbourhood $U$ of $c$ which is such that $U \cap e_j \subset e_j$ for each $j \in J_0$, and suppose that $\Psi = 1$ in a neighbourhood $U_0 \subset U$ of $c$. Then, for $F = IG \in \mathcal{D}(H_{\Gamma})$,

$$(H_{\Gamma} F, I \Psi)_\Gamma = \sum_{j \in J_0, \epsilon(e_j) = 1} \int_{U_\epsilon(e_j)} \frac{dG}{d\sigma} \frac{d\Psi}{d\sigma} \beta(\sigma) d\sigma \epsilon(e_j) G' = -\sum_{j \in J_0} \epsilon(e_j) G'(\sigma) \beta(\sigma) + \int_{U_0} \frac{dG}{d\sigma} \frac{d\Psi}{d\sigma} \beta(\sigma) d\sigma,$$

where $e_j \cap U_0 = (c, \sigma_j)$ if $\epsilon(e_j) = 1$ and $(\sigma, c)$ otherwise. Hence, on using (3.23),

$$(H_{\Gamma} F, I \Psi)_\Gamma = -\sum_{j \in J_0} \epsilon(e_j) G'(\sigma) \beta(\sigma) + (H_{\Gamma} F, I \Psi)_\Gamma + \int_{U_0} \frac{dG}{d\sigma} \beta G' \frac{d\Psi}{d\sigma} d\sigma.$$

On allowing $U_0$ to shrink to $c$, (c) follows on using (a) and

$$\left| \int_{U_0} \frac{dG}{d\sigma} \beta G' \frac{d\Psi}{d\sigma} d\sigma \right| \leq \left( \int_{U_0} \frac{1}{\alpha(\sigma)} \left| \frac{dG}{d\sigma} \beta G' \frac{d\Psi}{d\sigma} \right|^2 d\sigma \right)^{1/2} \left( \int_{U_0} \alpha(\sigma) |\Psi|^2 d\sigma \right)^{1/2} \to 0.$$

(II) Suppose that $F$ satisfies (a) ~ (c). Then set

$$F^*(\sigma) = -\frac{1}{\alpha(\sigma)} \frac{d}{d\sigma} \left( \beta(\sigma) \frac{dG}{d\sigma} \right),$$

where $G = I^{-1} F$. It follows that $F^* \in L^2(\Gamma)$. For $\Psi \in H^1(\Gamma)$, we have, by partial integration,

$$\sum_{j \in J} \int_{b_j} \beta(\sigma) \frac{dG(\sigma)}{d\sigma} \frac{d\Psi(\sigma)}{d\sigma} d\sigma = (F^*, \Psi)_\Gamma + \sum_{j \in J} \left[ \beta(\sigma) \frac{dG(\sigma)}{d\sigma} \frac{\Psi(\sigma)}{d\sigma} \bigg|_{\sigma = b_j}^{\sigma = a_j} \right].$$

Here, since $\psi$ is continuous on $\Gamma$, the condition (c) can be used to show that $C = 0$, which implies that

$$(G \cdot \psi)_1, \Gamma = (F^* + F, \psi)_\Gamma \quad (\psi \in H^1(\Gamma)).$$

Thus, by definition, $F \in \mathcal{D}(H_{\Gamma})$ and $F^* = H_{\Gamma} F$. This completes the proof.
4. The inclusion $\sigma_e(-\Delta_\Omega) \subseteq \sigma_e(H_T)$

Let $\sigma_e(-\Delta_\Omega)$ and $\sigma_e(H_T)$ denote, respectively, the essential spectra of the Neumann Laplacian $-\Delta_\Omega$ in $L^2(\Omega)$ and the self-adjoint operator $H_T$ defined in (3.21). Recall that the essential spectrum of a self-adjoint operator is the complement in its spectrum of the set of isolated eigenvalues of finite multiplicity. Equivalent descriptions of the essential spectrum are given in [3, Chapter IX]. We shall need in this section the following assumption: with $\alpha, \beta$ given by (2.10) and (3.12)(i), we set

$$\epsilon(\Gamma_0) := \sup_{e(\Gamma_0) \neq \emptyset} \sup_{\sigma(\epsilon) \in e} \alpha(\epsilon)\beta(\sigma)$$

for any measurable subset $\Gamma_0$ of $\Gamma$.

**Assumption 4.1.** There exists a sequence $\{\Gamma_m : m \in \mathbb{N}\}$ of non-empty subsets of $\Gamma$ which are such that

(i) $\Gamma_1 \subset \Gamma_2 \subset \ldots$,
(ii) $\Gamma = \bigcup_{m=1}^{\infty} \Gamma_m$,
(iii) the embeddings $E_m : H^1(\Omega) \hookrightarrow L^2(\Omega_m)$ are compact, where $\Omega_m = \tau^{-1}(\Gamma_m)$,
(iv) $\lim_{m \to \infty} \epsilon(\Gamma \setminus \Gamma_m) = 0$.

This is a mild assumption; indeed, parts (i)–(iii) are natural for any domain $\Omega$ since the $\Omega_m$ keep clear of any irregular part of the boundary of $\Omega$ which may cause its embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ to be non-compact.

**Lemma 4.2.** Suppose that Assumption 2.1 holds. Let $\Gamma_0$ be a measurable subset of $\Gamma$ and set $\Omega_0 = \tau^{-1}(\Gamma_0)$. Then we have

(i) for all $f \in L^2(\Omega)$

$$\int_{\Omega_0} |P_0 f(x)|^2 \, dx \leq \int_{\Omega_0} |f(x)|^2 \, dx \quad (f \in L^2(\Omega)), \quad (4.2)$$

(ii) for $f \in H^1(\Omega)$ and $g = Ef$, where $E : H^1(\Omega) \hookrightarrow L^2(\Omega)$,

$$\int_{\Omega_0} |(P_0 - 1)g(x)|^2 \, dx \leq \epsilon(\Gamma_0) \int_{\Omega_0} |\nabla f(x)|^2 \, dx. \quad (4.3)$$

**Proof.** (i) Since $P_0 = T_0^T T_0^*$, we have from (2.9) and (3.11)

$$\int_{\Omega_0} |P_0 f(x)|^2 \, dx$$

$$= \int_{\Omega_0} |(T_0^T f)(x)|^2 \alpha(\sigma) \, d\sigma$$

$$= \sum_{j \in J} \int_{\mathcal{L}_j} \alpha(\sigma)^{-1} \left| \int_{\mathcal{L}_j} f(x(\sigma, s)) \frac{ds}{|\nabla \tau(x(\sigma, s))|} \right|^2 \, d\sigma$$

$$\leq \sum_{j \in J} \int_{\mathcal{L}_j} \left| \int_{\mathcal{L}_j} |f(x(\sigma, s))|^2 \frac{ds}{|\nabla \tau(x(\sigma, s))|} \right| \, d\sigma$$

$$= \int_{\Omega_0} |f(x)|^2 \, dx.$$
(ii) Let \( g \in H^1(\Omega) \cap C^1(\Omega) \) and \( g = Ef \). Then, if \( x = x(\sigma, s) \), \( \tau(x) = t(\sigma) \in e \), we have from (3.11)

\[
(P_0 g)(x) - g(x) = (T_0 \gamma)(t(\sigma)) - g(x)
= \frac{1}{\alpha(\sigma)} \int_{\ell_+(\sigma)}^{\ell_-(\sigma)} g(\sigma, s') \frac{1}{|\nabla \tau(\sigma, s')|} ds' - g(\sigma, s)
= \frac{1}{\alpha(\sigma)} \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} \{g(\sigma, s') - g(\sigma, s)\} \frac{1}{|\nabla \tau(\sigma, s')|} ds'.
\]

(4.4)

Also, since \( s \) is the arc length of the curve \( x = x(\sigma, s) \), i.e., \( C(t(\sigma)) \), we have

\[
\left| \frac{\partial x(\sigma, s)}{\partial s} \right| = 1.
\]

Consequently

\[
\left| g(x(\sigma, s')) - g(x(\sigma, s)) \right|
= \left| \int_s^{s'} \nabla f(\sigma, \rho) \cdot \frac{\partial x(\sigma, \rho)}{\partial \rho} d\rho \right|
\leq \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla f(\sigma, \rho)| d\rho.
\leq \left( \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla f(\sigma, \rho)|^2 \frac{d\rho}{|\nabla \tau(\sigma, \rho)|} \right)^{1/2} \beta(\sigma)^{1/2}
\]

by Cauchy’s inequality and the definition of \( \beta \) given in (3.12)(i). From (4.4) it follows that

\[
\left| P_0 g(x) - g(x) \right| \leq \left( \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla f(\sigma, \rho)|^2 \frac{d\rho}{|\nabla \tau(\sigma, \rho)|} \right)^{1/2} \beta(\sigma)^{1/2}.
\]

On substituting in (2.7), with \( \{e_j : j \in J_0\} \) the set of edges which intersect \( \Gamma_0 \), we derive

\[
\int_{\Omega_0} \left| P_0 g(x) - g(x) \right|^2 dx \leq \sum_{j \in J_0} \int_{e_j} d\sigma \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} \beta(\sigma) \left( \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} \frac{|\nabla f(x(\sigma, \rho))|^2}{|\nabla \tau(x(\sigma, \rho))|} d\rho \right) d\sigma
\times |\nabla \tau(x(\sigma, s))|
= \sum_{j \in J_0} \int_{e_j} \alpha(\sigma) \beta(\sigma) \left( \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} \frac{|\nabla f(x(\sigma, \rho))|^2}{|\nabla \tau(x(\sigma, \rho))|} d\rho \right) d\sigma
\leq \epsilon(\Gamma_0) \int_{\Omega_0} |\nabla f(x)|^2 dx,
\]

which completes the proof of (ii).

The following lemma is an analogue of [1, Theorem 3.1].

**LEMMA 4.3.** Suppose that Assumptions 2.1 and 4.1 are satisfied. Then it follows that \( (P_0 - 1)E : H^1(\Omega) \to L^2(\Omega) \) is compact.
Proof. Let $\chi_m$ denote the characteristic function of $\Omega_m = \tau^{-1}(\Gamma_m)$. Then, from (4.2) and Assumption 4.1(iii), $\chi_m(P_0 - 1)E : H^1(\Omega) \to L^2(\Omega)$ is compact. Also

$$\|\{(P_0 - 1)(-\Delta + 1)^{-1/2} - \chi_m(P_0 - 1)(-\Delta + 1)^{-1/2}\}f\|_{\Omega}$$

$$= \|(1 - \chi_m)(P_0 - 1)E[-(\Delta + 1)^{-1/2}]f\|_{\Omega}$$

$$\leq \sqrt{e(\Gamma_1 m)}\|[-(\Delta + 1)^{-1/2}]f\|_{1, \Omega} \quad \text{(by (4.3))}$$

$$= \sqrt{e(\Gamma_1 m)}\|f\|_{\Omega} \quad \text{(by (3.6))}.$$

It follows that, as $m \to \infty$,

$$\|\{(P_0 - 1)(-\Delta + 1)^{-1/2} - \chi_m(P_0 - 1)(-\Delta + 1)^{-1/2}\}f\| \to 0,$$

where $\| \|$ denotes the operator norm from $L^2(\Omega)$ into $L^2(\Omega)$. Since

$$\chi_m(P_0 - 1)(-\Delta + 1)^{-1/2} = \chi_m(P_0 - 1)E[-(\Delta + 1)^{-1/2}]$$

is compact on $L^2(\Omega)$, the lemma is proved.

**Theorem 4.4.** Suppose that Assumption 2.1 holds, and assume that $E(P_1 - 1) : H^1(\Omega) \to L^2(\Omega)$ is compact. Then $\sigma_e(-\Delta) \subseteq \sigma_e(H_\Gamma)$.

**Proof.** We first prove that

$$E P_1 E^{-1}(-\Delta + 1)^{-1} = T_0(H_\Gamma + 1)^{-1} T_0^*.$$\hspace{1cm} (4.5)

In fact, from (3.3), (3.18), (3.19) and (3.22)

$$T_0(H_\Gamma + 1)^{-1} T_0^* = T_0 I I T_0^*$$

$$= E T_1 T_1^* E^*$$

$$= (E P_1 E^{-1}) (H_\Gamma + 1)^{-1},$$

and hence (4.5) is established. The second step is to prove

$$\sigma_e([-\Delta + 1]^{-1})[0] \subseteq \sigma_e(E P_1 E^{-1}(-\Delta + 1)^{-1})[0].$$\hspace{1cm} (4.6)

Let $\{g_n\}$ be a singular sequence of $(-\Delta + 1)^{-1}$ at $\lambda > 0$;

$$\|\{[-\Delta + 1]^{-1} - \lambda\}g_n\|_{\Omega} \to 0, \quad \|g_n\|_{\Omega} = 1, \quad g_n \to 0 \text{ in } L^2(\Omega)$$\hspace{1cm} (4.7)

as $n \to \infty$. Set

$$k_n = \frac{g_n}{\|(-\Delta + 1)^{-1/2}g_n\|_{\Omega}}, \quad h_n = (-\Delta + 1)^{-1/2} k_n.$$\hspace{1cm} (4.8)

Note that

$$1 \geq \|(-\Delta + 1)^{-1/2}g_n\|_{\Omega} \geq \|(-\Delta + 1)^{-1}g_n\|_{\Omega} \to \lambda > 0.$$
We have in $L^2(\Omega)$

$$\|(−Δ_\Omega + 1)^{-1} − λ\)k_n\|_\Omega → 0, \ k_n → 0, \ \|k_n\|_\Omega × 1,$$  

(4.9)

$$\|h_n\|_\Omega = 1, \ h_n → 0,$$  

(4.10)

and

$$\|EP_1E^{-1}(−Δ_\Omega + 1)^{-1} − λ\)h_n\|_\Omega = \|EP_1E^{-1}(−Δ_\Omega + 1)^{-1/2}(−Δ_\Omega + 1)^{-1} − λ\)k_n\|_\Omega → 0$$  

(4.11)

from (4.9) and since $EP_1E^{-1}(−Δ_\Omega + 1)^{-1/2}$ is bounded in $L^2(\Omega)$, by (3.5). To establish (4.6), it is therefore sufficient to prove that

$$\|(EP_1E^{-1} − 1)h_n\|_\Omega → 0.$$  

(4.12)

From (4.8), (4.9) and (3.5) $E^{-1}h_n = E^{-1}(−Δ_\Omega + 1)^{-1/2}k_n \rightarrow 0$ in $H^1(\Omega)$. Hence (4.12) will follow if

$$E(P_1 − 1) : H^1(\Omega) → L^2(\Omega)$$

is compact. Since this property is part of the hypothesis, the inclusion (4.6) is proved.

The final step is to prove

$$σ_e(T_0(HΓ + 1)^{-1}T_0^*\})\{0\} ⊆ σ_e((HΓ + 1)^{-1})\{0\}.$$  

(4.13)

On using (4.5), (4.6) and the Spectral Mapping Theorem, (4.13) will conclude the proof. Let $λ ∈ σ_e(T_0(HΓ + 1)^{-1}T_0^*)\{0\}$ with singular sequence \{ν_n\}:

$$\|(T_0(HΓ + 1)^{-1}T_0^* − λ\)v_n\|_Ω → 0, \ \|v_n\|_Ω = 1, \ v_n → 0 in L^2(Ω).$$  

(4.14)

Set

$$w_n = \frac{T_0(HΓ + 1)^{-1}T_0^*v_n}{\|T_0(HΓ + 1)^{-1}T_0^*v_n\|_Ω} =: T_0w_n.$$  

(4.15)

It is readily seen that \{w_n\} is also a singular sequence of $T_0(HΓ + 1)^{-1}T_0^*$ at $λ$. Moreover, since $T_0^*w_n = w_n$,

$$\|(HΓ + 1)^{-1} − λ\)W_n\|_Γ = \|T_0\|(HΓ + 1)^{-1} − λ\)W_n\|_Ω$$

$$= \|(T_0[HΓ + 1]^{-1}T_0^* − λ\)w_n\|_Ω → 0,$$

$$\|W_n\|_Γ = \|T_0W_n\|_Ω = \|w_n\|_Ω = 1$$

and, for all $G ∈ L^2(Γ)$,

$$(W_n, G)_Γ = (w_n, T_0G)_Ω → 0.$$

Hence \{w_n\} is a singular sequence of $(HΓ + 1)^{-1}$ at $λ$ and the proof is complete.
5. The reverse inclusion

The problem concerning the reverse inclusion to that in Theorem 4.4 is more difficult to handle. The reverse inclusion seems unlikely to be true in general, and we are forced to make a technical assumption to establish the main theorem in this section; this also yields a criterion for the compactness of $E(P_1 - 1) : H^1(\Omega) \to L^2(\Omega)$ needed in Theorem 4.4; see Corollary 5.3.

ASSUMPTION 5.1. (i) $\ell_\pm \in C^1(e)$, for each $e \in E(\Gamma)$, where $E(\Gamma)$ is the set of all edges of $\Gamma$;
(ii) $\tau \in C^2(\Omega \setminus \{\alpha(\Gamma) \cup \tau^{-1}(V(\Gamma))\})$, where $V(\Gamma)$ is the set of all vertices of $\Gamma$;
(iii) for $x = x(\sigma, s)$ and $\tau(x) = t(\sigma) \in e \in E(\Gamma)$, we have that $t(\sigma) = \sigma a$, where $a$ is the unit vector along $e$ and so $\sigma = \tau(x) \cdot a$; set

$$\delta(\sigma) := \left(\frac{\alpha(\sigma)}{\beta(\sigma)}\right)^{1/2} \sup_{-\ell_- \leq s \leq \ell_+} \left[\left|\nabla\sigma - \frac{\beta(\sigma)}{\alpha(\sigma)} \frac{\partial x(\sigma, s)}{\partial s}\right|^{-1}\right]$$

$$+ \frac{1}{2} \left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{3/2} \left[\int_{\ell_-}^{\ell_+} \left|\frac{\partial}{\partial s}(|\nabla\tau(\sigma, s)|)^{-1}\right| ds\right]$$

$$+ \frac{3}{2} \left|\int_{\ell_-}^{\ell_+} \left[\nabla\tau(\sigma, s)\right]^{-1} \left[\int_{\ell_-}^{\ell_+} \left|\nabla\tau(\sigma, s)\right|^{-1}\right]\right| ds\right] \left|\nabla\tau(\sigma, s)\right|^{-1}\right].$$

For a subtree $\Gamma_0$ of $\Gamma$, set

$$\theta(\Gamma_0) := \sup_{e \in E(\Gamma_0)} \sup_{\sigma \in e} \delta(\sigma).$$

Then $\theta(\Gamma) < \infty$.

The main result in this section rests heavily on the following analogue of [1, Proposition 2.1].

PROPOSITION 5.2. Suppose that Assumptions 2.1 and 5.1 are satisfied. Then, for $f \in \mathcal{R}(E)$, $T_0^a f$ is differentiable on $e$ for any $e \in E(\Gamma)$, and

$$\left(\frac{\beta}{\alpha}\right)^{1/2} \frac{d}{d\sigma}(T_0^a f) \in L^2(\Gamma).$$

Further, for $f = Eg$, $g \in H^1(\Omega) \cap C^1(\Omega)$, $G \in H^1(\Gamma)$, we have

$$\left|\int_{\Omega} \nabla g \cdot \nabla(T_0^a G) dx - \int_{\Gamma} \frac{d}{d\sigma}(T_0^a f)(\sigma) \frac{dG(\sigma)}{d\sigma} \beta(\sigma) d\sigma\right| \leq \delta(\cdot) \left(\frac{\beta}{\alpha}\right)^{1/2} \left\|\frac{dG(\cdot)}{d\sigma}\right\|_{L^2(\Gamma)} \left\|\nabla g\right\|_{L^2(\Omega)}.$$

where $\delta$ is defined in Assumption 5.1(iii). If $T_0^a f$ is continuous on $\Gamma$ then $T_0^a f \in IH^1(\Gamma)$.

Proof. The differentiability of $T_0^a f$ on each $e$ follows from Assumption 5.1. For $x = x(\sigma, s)$ and $\tau(x) = t(\sigma) = \sigma a$, where $a$ is the unit vector along $e \in E(\Gamma)$, we have

$$\nabla g \cdot \nabla(T_0^a G) = \nabla g \cdot \left(\frac{dG}{d\sigma}\right) = \nabla g \cdot \nabla(\sigma) \left(\frac{dG}{d\sigma}\right).$$
and from (2.9) we have
\[
\int_{\Omega} \nabla g \cdot \nabla (T_{ij} G) \, dx = \sum_{j \in J} \int_{\partial e_j} \frac{dG}{d\sigma} \, d\sigma \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} (\nabla g \cdot \nabla \sigma) \frac{ds}{|\nabla \tau(\sigma, s)|}.
\] (5.2)

From Lemma 3.1
\[
\frac{d}{d\sigma}(T_{ij} f)(\sigma) = -\frac{\alpha'(\sigma)}{\alpha(\sigma)^2} \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} f(\sigma, s) \frac{ds}{|\nabla \tau(\sigma, s)|} + \frac{d\ell_{+}(\sigma)}{d\sigma} \alpha(\sigma)^{-1} f(\sigma, \ell_{+}(\sigma))|\nabla \tau(\sigma, \ell_{+}(\sigma))|^{-1} - \frac{d\ell_{-}(\sigma)}{d\sigma} \alpha(\sigma)^{-1} g(\sigma, -\ell_{-}(\sigma))|\nabla \tau(\sigma, -\ell_{-}(\sigma))|^{-1} + \alpha(\sigma)^{-1} \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} \nabla g(\sigma, s) \cdot \frac{\partial}{\partial \sigma} \chi(\sigma, s) \frac{ds}{|\nabla \tau(\sigma, s)|}.
\]

Therefore, we have from (2.10)
\[
\alpha'(\sigma) = \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} \frac{\partial}{\partial \sigma} (|\nabla \tau(\sigma, s)|^{-1}) \, ds + \frac{\ell_{+}(\sigma)}{|\nabla \tau_{+}(\sigma)|} + \frac{\ell_{-}(\sigma)}{|\nabla \tau_{-}(\sigma)|}.
\] (5.4)

where \( \nabla \tau_{\pm}(\sigma) = \nabla \tau(\sigma, \pm \ell_{\pm}(\sigma)) \). On writing \( f_{\pm}(\sigma) = f(\sigma, \pm \ell_{\pm}(\sigma)) \), we have
\[
\alpha(f_2 + f_3) = \frac{\ell_{+} f_+}{|\nabla \tau_+|} + \frac{\ell_{-} f_-}{|\nabla \tau_-|}
= \frac{1}{2} \left\{ \left( \frac{\ell_{+}}{|\nabla \tau_+|} - \frac{\ell_{-}}{|\nabla \tau_-|} \right)(f_+ - f_-) + \left( \frac{\ell_{+}}{|\nabla \tau_+|} + \frac{\ell_{-}}{|\nabla \tau_-|} \right)(f_+ + f_-) \right\}
= \frac{1}{2} \left\{ \frac{\ell_{+}}{|\nabla \tau_+|} - \frac{\ell_{-}}{|\nabla \tau_-|} \right\}(f_+ - f_-) + \frac{\alpha'}{\alpha} \int_{-\ell_{-}}^{\ell_{+}} (f_+ + f_-) \frac{ds}{|\nabla \tau|}
- \int_{-\ell_{-}}^{\ell_{+}} (f_+ + f_-) \frac{\partial}{\partial \sigma} \left( |\nabla \tau|^{-1} \right) \, ds.
\]

Therefore,
\[
\alpha(\sigma)(f_1 + f_2 + f_3 + f_4 + f_5)(\sigma)
= -\frac{\alpha'(\sigma)}{\alpha(\sigma)} \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} \left\{ f(\sigma, s) - \frac{1}{2} (f_+ + f_-)(\sigma) \right\} \frac{ds}{|\nabla \tau(\sigma, s)|}
+ \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} \left\{ f(\sigma, s) - \frac{1}{2} (f_+ + f_-)(\sigma) \right\} \frac{\partial}{\partial \sigma} \left( |\nabla \tau(\sigma, s)|^{-1} \right) \, ds
+ \frac{1}{2} \left( \frac{\ell_{+}(\sigma)}{|\nabla \tau_{+}(\sigma)|} - \frac{\ell_{-}(\sigma)}{|\nabla \tau_{-}(\sigma)|} \right)(f_+ - f_-)(\sigma)
+ \int_{-\ell_{-}(\sigma)}^{\ell_{+}(\sigma)} \nabla g(\sigma, s) \cdot \frac{\partial}{\partial \sigma} \chi(\sigma, s) \frac{ds}{|\nabla \tau(\sigma, s)|}.
\] (5.5)
On substituting (5.5) in (5.3), we obtain
\[
\int_{\Omega} \nabla g \cdot \nabla (T_1 G) \, dx - \int_{\Gamma} \frac{d}{d\sigma} (T_0^* f)(\sigma) \frac{dG(\sigma)}{d\sigma} \beta(\sigma) \, d\sigma \\
= \sum_{j \in J} \int_{e_j} \frac{dG}{d\sigma} \left[ \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} (\nabla g \cdot \nabla \sigma)(\sigma, s) - \left( \frac{\beta(\sigma)}{\alpha(\sigma)} \frac{\partial}{\partial \sigma} \right)(\sigma, s) \right] \frac{ds}{|\nabla \tau(\sigma, s)|} \\
- \frac{\beta(\sigma)}{\alpha(\sigma)} \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \left\{ f(\sigma, s) - \frac{1}{2} (f_+ + f_-)(\sigma) \right\} \frac{\partial}{\partial \sigma} \left( |\nabla \tau(\sigma, s)|^{-1} \right) \frac{ds}{|\nabla \tau(\sigma, s)|} \\
+ \frac{\beta(\sigma) \alpha'(\sigma)}{\alpha(\sigma)^2} \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \left\{ f(\sigma, s) - \frac{1}{2} (f_+ + f_-)(\sigma) \right\} \frac{ds}{|\nabla \tau(\sigma, s)|} \\
- \frac{\beta(\sigma)}{2\alpha(\sigma)} \left( \frac{\ell_+(\sigma)}{|\nabla \tau(\sigma, s)|} - \frac{\ell_-(\sigma)}{|\nabla \tau(\sigma, s)|} \right) \left( f_+ - f_-)(\sigma) \right) \right].
\]
(5.6)

Also, since \( s \) is arc length along \( C(t) \), we have
\[
\frac{\partial x(\sigma, s)}{\partial s} = 1,
\]
and hence for \( s_1, s_2 \in C(t), \ t = t(\sigma) \),
\[
|f(\sigma, s_2) - f(\sigma, s_1)| = \left| \int_{s_1}^{s_2} \nabla g(\sigma, s) \frac{\partial}{\partial s} x(\sigma, s) \, ds \right| \\
\leq \int_{s_1}^{s_2} |\nabla g(\sigma, s)| \, |ds|.
\]
(5.7)

On using (5.4) and (5.7), the right-hand side of (5.6) is majorized by
\[
\sum_{j \in J} \int_{e_j} \frac{dG}{d\sigma} \left[ \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \nabla g(\sigma, s) \cdot \left( \nabla \sigma - \frac{\beta(\sigma)}{\alpha(\sigma)} \frac{\partial}{\partial \sigma} x(\sigma, s) \right) \frac{ds}{|\nabla \tau(\sigma, s)|} \\
+ \left( \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla g(\sigma, s)| \, ds \right) \frac{\beta(\sigma)}{\alpha(\sigma)} \left\{ 2 \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \left| \frac{\partial}{\partial \sigma} \left( |\nabla \tau(\sigma, s)|^{-1} \right) \right| \frac{ds}{|\nabla \tau(\sigma, s)|} \\
+ \frac{3}{2} \left( \frac{|\ell_+(\sigma)|}{|\nabla \tau(\sigma, s)|} + \frac{|\ell_-(\sigma)|}{|\nabla \tau(\sigma, s)|} \right) \right\} \right].
\]
(5.8)

Finally, on substituting
\[
\int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \left| \nabla g(\sigma, s) \cdot \left( \nabla \sigma - \frac{\beta(\sigma)}{\alpha(\sigma)} \frac{\partial}{\partial \sigma} x(\sigma, s) \right) \right| \frac{ds}{|\nabla \tau(\sigma, s)|} \\
\leq \left[ \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} |\nabla g(\sigma, s)|^2 \frac{ds}{|\nabla \tau(\sigma, s)|} \right]^{1/2} \\
\times \left[ \int_{-\ell_-(\sigma)}^{\ell_+(\sigma)} \left| \frac{\beta(\sigma)}{\alpha(\sigma)} \frac{\partial}{\partial \sigma} x(\sigma, s) \right|^2 \frac{ds}{|\nabla \tau(\sigma, s)|} \right]^{1/2}
\]
and
\[ \int_{\ell_+}^{\ell_-} |\nabla g(\sigma, s)| ds \leq \sqrt{\beta(\sigma)} \left[ \int_{\ell_+}^{\ell_-} |\nabla g(\sigma, s)|^2 \frac{ds}{|\nabla \tau(\sigma, s)|} \right]^{1/2} \]
in (5.8), we obtain
\[ \int_{\Omega} \nabla g \cdot \nabla(T \delta_{\Omega}) dx - \int_{\Gamma} \frac{d}{d\sigma} (T_0^* f)(\sigma) \frac{dG(\sigma)}{d\sigma} - \beta(\sigma) d\sigma \]
\[ \leq \sum_{j \notin J} \int_{\ell_+}^{\ell_-} \frac{dG(\sigma)}{d\sigma} \delta(\sigma) \sqrt{\beta(\sigma)} \left[ \int_{\ell_+}^{\ell_-} |\nabla g(\sigma, s)|^2 \frac{ds}{|\nabla \tau(\sigma, s)|} \right]^{1/2} d\sigma \]
\[ \leq \left[ \sum_{j \notin J} \int_{\ell_+}^{\ell_-} d\sigma \int_{\ell_+}^{\ell_-} |\nabla g(\sigma, s)|^2 \frac{ds}{|\nabla \tau(\sigma, s)|} \right]^{1/2} \left[ \sum_{j \notin J} \int_{\ell_+}^{\ell_-} \frac{dG(\sigma)}{d\sigma} \delta(\sigma)^2 \beta(\sigma) d\sigma \right]^{1/2} \]
\[ = \|\nabla g\|_{\Omega} \left[ \int_{\Gamma} \frac{dG(\sigma)}{d\sigma} \delta(\sigma)^2 \beta(\sigma) d\sigma \right]^{1/2} \]
by (2.9). The proposition is therefore proved.

**Corollary 5.3.** Suppose that Assumptions 2.1, 4.1 and 5.1 are satisfied and that \( R(T_0^* E) \subseteq R(I) \). Also, suppose that either \( E \) is compact or that the following hold:

(i) there exist subsets \( \Gamma_m \) of \( \Gamma \), \( m = 1, 2, \ldots \), \( \Psi_m \in C_0^1(\Gamma_m) \) such that with \( \Omega_m = \tau^{-1}(\Gamma_m) \), \( H^1(\Omega) \hookrightarrow L^2(\Omega_m) \) is compact,

\[ \Psi_m = 1 \quad \text{on} \quad \Gamma_{m-1} \quad \|\Psi_m\| \leq c_m, \]

and
\[ \lim_{m \to 0} \left( c_m^2 \sup_{\Gamma_m \cap \Gamma_{m-1}} \frac{\beta(\sigma)}{\alpha(\sigma)} \right) = 0; \]

(ii) \( \lim_{m \to 0} \theta(\Gamma \setminus \Gamma_m) = 0. \)

Then \( E(P_1 - 1) : H^1(\Omega) \to L^2(\Omega) \) is compact.

**Remark 5.4.** Note the observation in Proposition 5.2 that \( R(T_0^* E) \subseteq R(I) \) if \( T_0^* E \) is continuous for each \( g \in H^1(\Omega) \). In condition (i) above, \( c_m = O(1) \) would require \( \Gamma \) to contain a path of infinite length. This is the case in Corollary 5.5 below.

**Proof of Corollary 5.3.** From (5.1) we have for all \( f = Eg, \ g \in H^1(\Omega) \) and \( G \in H^1(\Gamma) \),
\[ |(T_1^* - I^{-1} T_0^* E)g, G_{1,1} | \leq \left\| \delta(\cdot) \left( \frac{\beta(\cdot)}{\alpha(\cdot)} \right) \right\|^{1/2} \left\| \frac{dG(\cdot)}{d\sigma} \right\|_{\Gamma} \|\nabla g\|_{\Omega}. \quad (5.9) \]

Also, note that \((P_0 - 1) E : H^1(\Omega) \to L^2(\Omega) \) is compact by Lemma 4.3. Hence the compactness of \( E(P_1 - 1) \) is equivalent to the compactness of
\[ E P_1 - P_0 E = ET_1 T_1^* - T_0 T_0^* E \]
\[ = T_0 I - T_0 T_0^* E \quad \text{(by (3.19))} \]
\[ = T_0 I (T_1^* - I^{-1} T_0^* E) \quad (5.10) \]
since $\mathcal{R}(T^*_0 E) \subseteq \mathcal{R}(I)$ is assumed. Assumption 4.1(iii) implies that $I\Psi_m : H^1(\Gamma) \to L^2(\Gamma)$ is compact. For by (3.19), $I = T^*_0 ET_1$ and, with $\Omega_m = \tau^{-1}(\Gamma_m)$ and $G \in H^1(\Gamma)$

$$
\|T^*_0 ET_1 \Psi_m G\|^2_{L^2(\Omega_m)}
$$

by (2.9), which yields the compactness of $I\Psi_m$ since $\chi_{\Omega_m} ET_1 : H^1(\Gamma) \to L^2(\Omega_m)$ is compact. Hence, since $T^*_0 \rightarrow T^*_0 E : H^1(\Omega) \to H^1(\Gamma)$ is bounded by (5.9),

$$
T_0 I \Psi_m (T^*_0 - I^{-1} T^*_0 E) : H^1(\Omega) \to L^2(\Omega)
$$

is compact. Next, we use (5.9) to get

$$
\|(1 - \Psi_m \|T^*_1 - I^{-1} T^*_0 E) g, G\|_{1, \Gamma} \leq \|(1 - \Psi_m \|T^*_1 - I^{-1} T^*_0 E) g, [1 - \Psi_m G]\|_{1, \Gamma} + c_m \sup_{\Gamma_m \Gamma_m-1} \left(\beta(\sigma)\alpha(\sigma)\right)^{1/2} \|G\|_{1, \Gamma} \|G\|_{1, \Gamma}.
$$

Hence

$$
\|T_0 I (1 - \Psi_m) (T^*_1 - I^{-1} T^*_0 E)\|
$$

and the proof is complete.

**Corollary 5.5.** Suppose Assumptions 2.1, 4.1 and 5.1 are satisfied. Then there exists a bounded linear operator $K : H^1(\Gamma) \to H^1(\Omega)$ such that for $f \in \mathcal{R}(E)$ and $G \in H^1(\Gamma)$

$$
\int_{\Omega} \nabla g \cdot \nabla(T_1 G) dx - \int_{\Gamma} d\sigma \left(\frac{\partial}{\partial \sigma} (T^*_0 f)(\sigma)\frac{dG(\sigma)}{d\sigma}\beta(\sigma) d\sigma\right) = (g, KG)_{1, \Omega}.
$$

(5.11)
where \( f = Eg, \ g \in H^1(\Omega). \) Moreover, if there exists \( \Psi_m \in C^1_0(\Gamma_m), \ m = 1, 2, \ldots \) which is such that \( \Psi_m : \Gamma_m \to [0, 1], \ |\Psi_m| \leq 1 \) and \( \Psi_m = 1 \) on \( \Gamma_m' \subset \Gamma_m, \) then \( K\Psi_m I^{-1} H_1 + 1)^{-1} : L^2(\Gamma) \to H^1(\Omega) \) is compact and

\[
\| K (1 - \Psi_m) I^{-1} (H_1 + 1)^{-1} \| \leq c_m (\Gamma \setminus \Gamma_m')
\]

with positive constant \( c_m \) satisfying

\[
c_m^2 \leq 2 \max \left( 1, \max_{\Gamma_m \setminus \Gamma_m'} \left[ \beta(\sigma)/\alpha(\sigma) \right] \right).
\]

where the norm on the left-hand side is the operator norm from \( L^2(\Gamma) \) into \( H^1(\Omega), \) and \( \theta(\cdot) \) is defined in Assumption 5.1(iii). In particular, \( K I^{-1} (H_1 + 1)^{-1} : L^2(\Gamma) \to H^1(\Omega) \) is compact if \( c_m = O(1) \) and \( \lim_{m \to \infty} \theta(\Gamma \setminus \Gamma_m') = 0. \)

**Proof.** Since \( H^1(\Omega) \cap C^4(\Omega) \) is dense in \( H^1(\Omega), \) it follows from Proposition 5.2 that for \( G \in H^1(\Gamma), \) the left-hand side of (5.9) extends uniquely to a bounded linear functional on \( H^1(\Omega), \) and hence the existence of \( K \) is established. Moreover, from (5.1)

\[
\| KG \|_{1, \Omega} \leq \left\| \delta(\cdot) \left( \frac{\beta}{\alpha} \right)^{1/2} dG \right\|_\Gamma \cdot
\]

As in the proof of Corollary 5.3, Assumption 4.1(iii) implies that \( I\Psi_m : H^1(\Gamma) \to L^2(\Gamma) \) is compact. This in turn implies that \( I\Psi_m K^* : H^1(\Omega) \to L^2(\Gamma) \) is compact, and hence so is its adjoint \( K\Psi_m I^* = K\Psi_m I^{-1} (H_1 + 1)^{-1}. \) Also, from (5.12) and Assumption 5.1(iii)

\[
\| K (1 - \Psi_m) I^* G \|_{1, \Omega} \leq \left\| \delta(\cdot) (\beta \alpha^{-1})^{1/2} \left\| (1 - \Psi_m) I^* G \right\|_\Gamma \right.
\]

\[
\leq c_m \theta(\Gamma \setminus \Gamma_m') \| I^* G \|_{1, \Gamma}
\]

\[
\leq c_m \theta(\Gamma \setminus \Gamma_m') \| G \|_{1, \Gamma}
\]

with \( c_m \) given above. The rest of the assertion now follows easily.

**Lemma 5.6.** Suppose Assumptions 2.1 and 5.1 hold. Then for all \( g \in H^1(\Omega) \) and \( G \in \mathcal{D}(H_\Gamma), \) the limit

\[
A(g, G) := \lim_{m \to \infty} \sum_{e \in E(\Gamma_m), e = (c, d)} \left\{ (T_0^+ f)(d) \frac{d}{d\sigma} H(d) \beta(d) - (T_0^+ g)(c) \frac{d}{d\sigma} H(c) \beta(c) \right\}
\]

exists, where \( f = Eg \) and \( G = IH; \) recall that in \( e = (c, d) \in E(\Gamma_m), \) we have \( d \geq a < c \) with respect to the root \( a. \) Also

\[
|A(g, G)| \leq C \| g \|_{1, \Omega}(\| H_\Gamma G \|_\Gamma + \| H \|_{1, \Gamma})
\]

for some positive constant \( C. \) Therefore, there exists a linear operator \( A : \mathcal{D}(H_\Gamma) \to H^1(\Omega) \) such that

\[
A(g, G) = (g, AG)_{1, \Omega}, \quad (g \in H^1(\Omega), G \in \mathcal{D}(H_\Gamma))
\]

and \( A(H_\Gamma + 1)^{-1} : L^2(\Gamma) \to H^1(\Omega) \) is bounded.
Proof. In (5.11) set
\[
\mathcal{A}_m(g, H) = \sum_{e \in E(\Gamma_m), e = (c, d)} \left\{ (T_0^* f)(d) \frac{d}{d\sigma} H(d) \beta(d) - (T_0^* f)(c) \frac{d}{d\sigma} H(c) \beta(c) \right\}.
\]
Then, from Theorem 3.3(d), with \(\Omega_m = \tau^{-1}(\Gamma_m)\),
\[
\int_{\Omega_m} f T_0^* (H G) dx = (f, T_0 H G)_{\Omega_m}
= (T_0^* f, H G)_{\Gamma_m}
= -\mathcal{A}_m(g, H) + \int_{\Gamma_m} \frac{d}{d\sigma} (T_0^* f) \frac{dH}{d\sigma} \beta(\sigma) d\sigma,
\]
and, on using Proposition 5.2, this yields the limit
\[
\mathcal{A}(g, G) = -\int_{\Omega} f T_0^* (H G) dx + \int_{\Gamma} \frac{d}{d\sigma} (T_0^* f) \frac{dH}{d\sigma} \beta(\sigma) d\sigma
= -(T_0^* f, H G)_{\Gamma}
- \left\{ \int_{\Omega} \nabla g \cdot \nabla(T_1 H) dx - \int_{\Gamma} \frac{d}{d\sigma} (T_0^* f) \frac{dH}{d\sigma} \beta(\sigma) d\sigma \right\}
+ \int_{\Omega} \nabla g \cdot \nabla(T_1 H) dx. \tag{5.16}
\]
Hence, on using \(T_0^* f = T_0^* E g\), and (5.1), we have, for \(g \in H^1(\Omega) \cap C^1(\Omega)\),
\[
|\mathcal{A}(g, G)| \leq \|g\|_{1,\Omega} \|H G\|_{\Gamma} + C \|H\|_{1,\Gamma} \|g\|_{1,\Omega} + \|g\|_{1,\Omega} \|H\|_{1,\Gamma}.
\]
Therefore \(\mathcal{A}(\cdot, G)\) can be extended to a bounded linear functional on \(H^1(\Omega)\), for each \(G \in \mathcal{D}(H_G)\).
It follows that (5.12) and (5.13) are satisfied. Also
\[
\|A G\|_{1,\Omega} \leq C \{\|H_G\|_{\Gamma} + \|H\|_{1,\Gamma}\},
\]
and thus, for all \(F \in L^2(\Gamma)\), since \(I^{-1}(H_G + 1)^{-1} = I^*\),
\[
\|A(H_G + 1)^{-1} F\|_{1,\Omega} \leq C \{\|F\|_{\Gamma} + \|I^* F\|_{1,\Gamma}\} \leq C \|F\|_{\Gamma}.
\]
The lemma is therefore proved.

Remark 5.7. If \(T_0^* f \in L^1(\Gamma)\) in Lemma 5.6, we have, on using the definition of \(H_G\),
\[
\mathcal{A}(g, G) = -\int_{\Omega} f T_0^* (H G) dx + \int_{\Gamma} \frac{d}{d\sigma} (T_0^* f) \frac{dH}{d\sigma} \beta(\sigma) d\sigma
= -(T_0^* f, H_G)_{\Gamma} + (T_0^* f, H_G)_{\Gamma}
= 0,
\]
However, in general \(T_0^* f\) may not be continuous on \(\Gamma\); see Example 6.2.2.
LEMMA 5.8. Suppose that Assumptions 2.1 and 5.1 are satisfied, and let $K$. A be the operators in Corollary 5.3 and Lemma 5.4. Then $T_0(H_T + 1)^{-1} = (-\Delta_\Omega + 1)^{-1}T_0$ is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$ if and only if $E(K^{-1} + A)(H_T + 1)^{-1} : L^2(\Gamma) \to L^2(\Omega)$ is compact.

Proof. From (5.16) and (5.11), for $g \in H^1(\Omega)$, $f = Eg$ and $G = IH \in \mathcal{D}(H_T)$,

$$-(g, AG)_{1, \Omega} = (f, T_0 H_T G)_{1, \Omega} + (g, KH)_{1, \Omega} - (g, T_I H)_{1, \Omega} + (Eg, ET_I H)_{\Omega}$$

and this gives, since $ET_I H = T_0 IH = T_0 G$,

$$(f, T_0(H_T + 1)G)_{1, \Omega} = (g, (T_I - AI - K)H)_{1, \Omega}.$$ 

It follows that $E(T_I - AI - K)H \in \mathcal{D}(-\Delta_\Omega)$ and

$$(-\Delta_\Omega + 1)[E(T_I - AI - K)H] = T_0(H_T + 1)G.$$ 

On setting $G = (H_T + 1)^{-1} \phi$, we have

$$(T_0 - EA - EKI^{-1}) (H_T + 1)^{-1} \phi = (-\Delta_\Omega + 1)^{-1} T_0 \phi,$$ 

and hence

$$T_0(H_T + 1)^{-1} = (-\Delta_\Omega + 1)^{-1} T_0 = E(A + KI^{-1})(H_T + 1)^{-1}.$$ 

(5.17)

This completes the proof.

COROLLARY 5.9. Suppose that the hypothesis of Theorem 4.4 holds, that Assumption 5.1 is satisfied, and that the operator $E(A + KI^{-1})(H_T + 1)^{-1} : L^2(\Gamma) \to L^2(\Omega)$ is compact. Then $\sigma_c(H_T) = \sigma_c(-\Delta_\Omega)$.

Proof. In view of Theorem 4.4 we have only to prove that $\sigma_c((H_T + 1)^{-1}) \setminus \{0\} \subset \sigma_c((-\Delta_\Omega + 1)^{-1}) \setminus \{0\}$. Let $\lambda \in \sigma_c((H_T + 1)^{-1}) \setminus \{0\}$ with singular sequence $\{F_n\}$. Then from (5.17), with $f_n = T_0 F_n$

$$(-\Delta_\Omega + 1)^{-1} f_n - \lambda f_n = T_0[(H_T + 1)^{-1} - \lambda] F_n - E(A + KI^{-1})(H_T + 1)^{-1} F_n.$$ 

It follows from this and the compactness of the operator $E(K^{-1} + A)(H_T + 1)^{-1}$ that $\{f_n\}$ is a singular sequence for $(-\Delta_\Omega + 1)^{-1}$ at $\lambda$.

COROLLARY 5.10. Suppose that the hypothesis of Theorem 4.4 and Assumption 5.1 are satisfied. Then we have $\sigma_c(-\Delta_\Omega) = \emptyset$ if and only if $\sigma_c(H_T) = \emptyset$.

Proof. Since $(-\Delta_\Omega + 1)^{-1} = EE^*$, it follows that $\sigma_c(-\Delta_\Omega) = \emptyset$ if and only if $E$ is compact, and the corollary is therefore a consequence of Corollary 5.9 and Theorem 4.4.

LEMMA 5.11. Suppose that Assumptions 2.1, 4.1 and 5.1 are satisfied. Let $\{F_n\}$ be a singular sequence for $(H_T + 1)^{-1}$ at $\lambda$, and set $f_n = T_0 F_n$. Then,

$$\lim_{m \to \infty} \|(-\Delta_\Omega + 1)^{-1} f_n - \lambda f_n\|_\Omega \leq \lim_{m \to \infty} \|E(A + KI^{-1})(H_T + 1)^{-1}\| \cdot \chi_m,$$ 

(5.18)

where, on the right-hand side of (5.18), $\| \cdot \|$ denotes the operator norm of $L^2(\Gamma)$ into $L^2(\Omega)$, and $\chi_m$ is the characteristic function of $\Omega_m = \tau^{-1}(\Gamma_m)$. 

Proof. From (5.17)
\[ (-\Delta_{\Omega} + 1)^{-1} f_n - \lambda f_n = T_0[(H_{\Gamma} + 1)^{-1} - \lambda]F_n - E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} F_n, \]
and thus, since the first term of the right-hand side of the above relation tends to 0 as \( n \to \infty, \)
\[ \lim_{n \to \infty} \|(-\Delta_{\Omega} + 1)^{-1} f_n - \lambda f_n \|_\Omega \leq \lim_{n \to \infty} \|E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} F_n \|. \] (5.19)
For (5.18), we write
\[ E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} = (1 - \chi_m)E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} + \chi_m E(A + K I^{-1})(H_{\Gamma} + 1)^{-1}, \]
and note that Assumption 4.1(iii) implies that \( \chi_m E : H^1(\Omega) \to L^2(\Omega) \) is compact. We then have
\[ \lim_{n \to \infty} \|(-\Delta_{\Omega} + 1)^{-1} f_n - \lambda f_n \|_\Omega \leq \|(1 - \chi_m)E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} \|, \] (5.20)
and hence (5.18).

**THEOREM 5.12.** Suppose that Assumptions 2.1, 4.1 and 5.1 are satisfied. Then
\[ \sigma_e((H_{\Gamma} + 1)^{-1}) \subseteq \{ \mu : \text{dist}(\mu, \sigma((-\Delta_{\Omega} + 1)^{-1})) \leq k \}, \] (5.21)
where
\[ k := \lim_{m \to \infty} \|(1 - \chi_m)E(A + K I^{-1})(H_{\Gamma} + 1)^{-1} \|. \] (5.22)

**Proof.** In (5.20) \( \{ f_n \} \) satisfies
\[ \|f_n\|_\Omega = 1, \quad f_n \rightharpoonup 0 \text{ in } L^2(\Omega) \] (5.23)
and \( \lambda \in \sigma_e((H_{\Gamma} + 1)^{-1}) \). Suppose that, contrary to (5.21),
\[ [\lambda - k - \epsilon, \lambda + k + \epsilon] \cap \sigma((-\Delta_{\Omega} + 1)^{-1}) = \emptyset \]
for some \( \epsilon > 0 \). Then, \( [\lambda - k - \epsilon, \lambda + k + \epsilon] \cap \sigma((-\Delta_{\Omega} + 1)^{-1}) \) contains at most a finite number of eigenvalues of \( (-\Delta_{\Omega} + 1)^{-1} \) of finite multiplicity; denote these by \( \lambda_j, j = 1, 2, \ldots, N \) and the corresponding eigenvectors by \( \phi_j \), the eigenvalues being repeated according to their multiplicity. Denoting the spectral function of \( (-\Delta_{\Omega} + 1)^{-1} \) by \( \mathcal{E}(\cdot) \) and \( (-\Delta_{\Omega} + 1)^{-1} \) by \( H_{\Omega} \), we have
\[ \| (H_{\Omega} - \lambda) f_n \|_\Omega^2 = \left( \int_{|\mu - \lambda| \leq k + \epsilon} + \int_{|\mu - \lambda| > k + \epsilon} \right) \| \mu - \lambda \|^2 \mathcal{E}(\mu) f_n, f_n \|_\Omega \]
\[ \geq \sum_{j=1}^{N} \| \lambda_j - \lambda \|^2 \| (f_n, \phi_j)_{\Omega} \|^2 + (k + \epsilon)^2 \int_{|\mu - \lambda| > k + \epsilon} \mathcal{E}(\mu) f_n, f_n \|_\Omega \]
\[ = \sum_{j=1}^{N} \| \lambda_j - \lambda \|^2 \| (f_n, \phi_j)_{\Omega} \|^2 + (k + \epsilon)^2 \left[ 1 - \sum_{j=1}^{N} \| (f_n, \phi_j)_{\Omega} \|^2 \right] \]
\[ = \sum_{j=1}^{N} \left[ \| \lambda_j - \lambda \|^2 - (k + \epsilon)^2 \right] \| (f_n, \phi_j)_{\Omega} \|^2 + (k + \epsilon)^2. \]
Since \( f_n \rightharpoonup 0 \text{ in } L^2(\Omega) \), it follows that
\[ \lim_{n \to \infty} \| (H_{\Omega} - \lambda) f_n \|_\Omega^2 \geq (k + \epsilon)^2 \]
which contradicts (5.16). The theorem is thus proved.
6. Examples

6.1 Horn-shaped domains

Let \( \Omega \) be the domain in Example 2.2, namely
\[
\Omega := \{ x = (x_1, x_2) : x_1 > 0, -\ell_-(x_1) < x_2 < \ell_+(x_1) \},
\]
with \( \Gamma = (0, \infty) \), and \( \tau(x_1, x_2) = x_1 \). Also suppose that \( \ell_\pm \in C^1((0, \infty)) \) and
\[
\begin{align*}
(i) \quad & \lim_{t \to \infty} \ell_\pm(t) = 0, \\
(ii) \quad & \lim_{t \to \infty} \ell_\pm'(t) = 0.
\end{align*}
\]
(6.1)

In this case, \( \alpha = \beta = \ell_+ + \ell_- \), and \( L^2(\Gamma), H^1(\Gamma) \) are the weighted Hilbert spaces.

\[
\| F \|_{\Gamma} = \left( \int_0^\infty |F(t)|^2 \alpha(t) \, dt \right)^{1/2},
\]
(6.2)

and
\[
\| F \|_{1,\Gamma} = \left( \int_0^\infty (|F'(t)|^2 + |F(t)|^2) \alpha(t) \, dt \right)^{1/2}.
\]
(6.3)

The operator \( H_{\Gamma} \) is given by
\[
H_{\Gamma} F = -\frac{1}{\alpha} (\alpha F')',
\]
(6.4)

with domain
\[
\mathcal{D}(H_{\Gamma}) = \{ F : F, \alpha F' \in AC_{\text{loc}}(0, \infty), F, H_{\Gamma} F \in L^2(\Gamma), (\alpha F')(0) = 0 \}.
\]
(6.5)

Assumption 4.1 is satisfied by (6.1)(i) and since \( \Omega \) has a smooth boundary. Also, in Assumption 5.1
\[
\delta(\sigma) = \frac{3}{2} (|\ell_+'(\sigma)| + |\ell_-'(\sigma)|).
\]

Hence (6.1)(ii) implies that \( KI^{-1}(H_{\Gamma} + 1)^{-1} : L^2(\Gamma) \to H^1(\Omega) \) is compact on account of Corollary 5.3. We also have
\[
(T_0^* g)(\sigma) = \frac{1}{\alpha(\sigma)} \int_{\ell_-(\sigma)}^{\ell_+(\sigma)} g(\sigma, s) \, ds.
\]

It follows from Proposition 5.2 and Remark 5.5 that, for \( g \in H^1(\Omega), T_0^* g \in IH^1(\Gamma) \) and \( A(g, G) = 0 \) in (5.11) for any \( G \in \mathcal{D}(H_{\Gamma}) \). Thus \( A \) is the zero operator and we conclude from Theorem 5.10 that
\[
\sigma_e(-\Delta_\Omega) = \sigma_e(H_{\Gamma}).
\]
(6.6)

The essential spectrum of \( H_{\Gamma} \) can be determined as follows (c.f. [1, Section 1]); note that it is
proved in [4, Corollary 5.2 and Section 6.3] that

\[(i) \sigma_e(-\Delta_{\Omega}) = 0 \iff \lim_{t \to \infty} \left( \int_0^t (\alpha(\sigma)^{-1} d\sigma) \left( \int_0^\infty \alpha(\sigma) d\sigma \right) \right) = 0, \]

\[(ii) 0 \in \sigma_e(-\Delta_{\Omega}) \iff \text{either } \int_0^{\infty} \alpha(\sigma) d\sigma = \infty \text{ or } \lim_{t \to \infty} \left( \int_0^t (\alpha(\sigma)^{-1} d\sigma) \left( \int_t^{\infty} \alpha(\sigma) d\sigma \right) \right) = \infty.\]

Define the operator \( T \) on \( L^2(0, \infty) \) by \( T = W^{-1}H_{\Gamma}^*W \), where \( W \) is the unitary operator from \( L^2(0, \infty) \) onto \( L^2(\Gamma) = L^2(\Gamma; \alpha dt) \) given by \( Wf = \alpha^{-1/2}f \).

Then \( H_{\Gamma} \) and \( T \) have the same essential spectra and for \( f \in D(T) \)

\[ Tf = -f'' + Vf, \]

where

\[ V(t) = \frac{1}{2} \left[ \frac{\ell''(t)}{\ell(t)} - \frac{\ell'(t)^2}{\ell(t)^2} \right] \quad (\ell(t) = \ell_+(t) + \ell_-(t)). \]

It follows that if

\[ \lim_{t \to \infty} V(t) = \eta \]

then \( \sigma_e(T) = [\eta, \infty) \) and so

\[ \sigma_e(-\Delta_{\Omega}) = \sigma_e(H_{\Gamma}) = [\eta, \infty). \]

The essential spectrum of \( H_{\Gamma} \) can also be derived from [8, Section 34].

### 6.2 Rooms and passages

#### 6.2.1

Let \( \tau \) be the map in Example 2.3 which was used in [4] to describe \( \Omega \) as a generalized ridged domain, with generalized ridge \([0, b]\), where \( b = \sum_{k=1}^{\infty} h_k \leq \infty \). The tree \( \Gamma \) consists of the vertices

\[ H_{2k-1}, H_{2k}, H_{2k} + \frac{1}{2}h_{2k+1} - \frac{1}{2}\delta_{2k}, H_{2k} + \frac{1}{2}h_{2k+1} + \frac{1}{2}\delta_{2k+2} \]

for \( k = 1, 2, \ldots \) and edges the interconnecting intervals; also \( u(t) = t \) on \([0, b]\).

In the typical portion of \( \Omega \) in Fig. 5 composed of a passage \( P \) of width \( \delta \) followed by a room \( R \) of side \( h \), the vertices are at the points \( O_i, i = 1, 2, \ldots, 5 \) and

\[ \tau(x_1, x_2) = \begin{cases} 
 x_1 & \text{in } P, \\
 \max \left( x_1, |x_2| - \frac{1}{2}\delta \right), 0 \leq x_1 \leq \frac{1}{2}h & \text{in } R
\end{cases} \]

and similarly in the rest of \( R \); note that we measure \( x_1 \) in \( R \) from \( O_2 \). The following are readily verified:
In $P$:
\[
\ell_{\pm} = \frac{1}{2} \delta, \quad \alpha = \beta = \ell_{+} + \ell_{-} = \delta,
\]
\[
(T_{0}^* g)(\sigma) = \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} g(\sigma, s) \, ds.
\]

In $R$:
\[
\alpha = \beta = \ell_{+} + \ell_{-},
\]
\[
\ell_{\pm}(\sigma) = \begin{cases} 
2\sigma + \frac{1}{2} \delta, & 0 \leq \sigma \leq \frac{1}{2} h - \frac{1}{2} \delta, \\
\frac{1}{2} h, & \frac{1}{2} h - \frac{1}{2} \delta \leq \sigma \leq \frac{1}{2} h,
\end{cases}
\]
\[
(T_{0}^* g)(\sigma) = \frac{1}{\alpha} \int_{-\ell_{-}}^{\ell_{+}} g(\sigma, s) \, ds,
\]

and similarly in the rest of $R$.

Assumption 4.1 is satisfied if
\[
\lim_{k \to \infty} h_k = 0
\]

since $\alpha(\sigma) \beta(\sigma) = O(h_{2k+1})$ in $P_{2k} \cup R_{2k+1}$. The operator $H_{\Gamma}$ is given by
\[
(H_{\Gamma} F)(\sigma) = -\frac{1}{\alpha(\sigma)} \frac{d}{d\sigma} \left( \alpha(\sigma) \frac{d F}{d\sigma} \right)
\]

with, in $P_{2k} \cup R_{2k+1},$
\[
\alpha(\sigma) = \begin{cases} 
\delta_{2k} & \text{in } P_{2k}, \\
2\sigma + \frac{1}{2} \delta_{2k} & \text{in } R_{2k+1}, \quad 0 \leq \sigma \leq \frac{1}{2} h_{2k+1} - \frac{1}{2} \delta_{2k}, \\
\frac{1}{2} h_{2k+1} & \text{in } R_{2k+1}, \quad \frac{1}{2} h_{2k+1} - \frac{1}{2} \delta_{2k} < \sigma < \frac{1}{2} h_{2k+1} - \frac{1}{2} \delta_{2k+2}, \\
2(h_{2k+1} - \sigma) + \frac{1}{2} \delta_{2k+2} & \text{in } R_{2k+1}, \quad \frac{1}{2} h_{2k+1} - \frac{1}{2} \delta_{2k+2} \leq \sigma \leq h_{2k+1},
\end{cases}
\]
and subject to the boundary conditions in Theorem 3.3(d) at the vertices. In Assumption 5.1, we have in Fig. 5

\[
\delta(\sigma) = \begin{cases} 
0 & \text{in } P, \\
6 & \text{in } R, 0 \leq \sigma \leq \frac{1}{2}h - \frac{1}{2}\delta, \\
0 & \text{in } R, \frac{1}{2}h - \frac{1}{2}\delta \leq \sigma \leq \frac{1}{2}h.
\end{cases}
\tag{6.13}
\]

Hence the condition \(\lim_{m \to \infty} \theta(\Gamma \setminus \Gamma_m) = 0\) required in Corollary 5.5 and 5.6 is not satisfied.

6.2.2 Suppose next we make the choice

\[
\tau(x_1, x_2) = x_1, (x_1, x_2) \in \Omega
\]

and \(\Gamma\) is the tree with vertices at \(H_k, k = 1, 2, \ldots\). Then

\[
\alpha(\sigma) = \beta(\sigma) = \ell_+(\sigma) + \ell_-(\sigma) = \begin{cases} 
\delta_{2k} & \text{in } P_{2k}, \\
\delta_{h_{2k+1}} & \text{in } R_{2k+1}.
\end{cases}
\]

and so, Assumption 4.1 is satisfied if (6.10) holds. Also, it is readily verified that, in Assumption 5.1, \(\delta(\sigma) = 0\) for all \(\sigma\) and hence, by (5.1) and (5.9), \(K\) is the zero operator. However,

\[
(T_0^* g)(\sigma) = \frac{1}{\alpha(\sigma)} \int_{-\alpha/2}^{\alpha/2} g(\sigma, s) \, ds
\]

which is not continuous at the vertex \(H_k\). Thus, we cannot claim that \(\mathcal{R}(T_0^* E) \subseteq \mathcal{R}(I)\), and we are forced to estimate the term \(A(g, G)\) in (5.13); these estimates are too crude to get \(k < 1\) in Theorem 5.12.

The next example indicates what is required to give \(\sigma_e(-\Delta_\Omega) = \sigma_e(\Gamma)\).

6.3 Modified rooms and passages

Let \(\Omega = \bigcup_{k=1}^{\infty} (P_{2k} \cup R_{2k+1})\), where

\[
P_{2k} = [H_{2k+1}, H_{2k}] \times (-\delta_{2k}/2, \delta_{2k}/2)
\]
and
\[ R_{2k+1} = \{(x_1, x_2) / -\ell_{+}^{(2k+1)}(x_1) < x_2 < \ell_{+}^{(2k+1)}(x_1), \ H_{2k} \leq x_1 \leq H_{2k+1}\}, \]
and \( \delta_{2k}, \ h_{2k+1} \) are as in Example 2.3; suppose that we allow \( \sum_{k=1}^{\infty} h_k = \infty \), and let \( H_k = \sum_{j=1}^{k} h_j \). Also we assume that \( \ell_{\pm}^{(2k+1)} \in C^1[H_{2k}, H_{2k+1}] \) and
\[
\begin{aligned}
\lim_{x_1 \to H_{2k}+} \ell_{+}^{(2k+1)}(x_1) &= \delta_{2k}, \\
\lim_{x_1 \to H_{2k+1}-} \ell_{-}^{(2k+1)}(x_1) &= \delta_{2k+2}.
\end{aligned}
\] (6.14)
The tree \( \Gamma \) has vertices at \( H_k, k = 0, 1, \ldots (H_0 = 0) \) and again \( u(t) = t \).

Define \( \tau(x_1, x_2) = x_1 \) throughout \( \Omega \). In the typical portion \( P \cup R \) illustrated in Fig. 6 we have \( \alpha = \beta = \ell_+ + \ell_- \). The conditions (6.14) imply that \( \alpha \) is continuous at the vertices and that \( T_0 g \in \mathcal{H}^1(\Gamma) \) for \( g \in \mathcal{H}^1(\Omega) \). Hence, by Remark 5.5, \( A = 0 \) and so \( A \) is the zero operator. Moreover, Assumption 4.1 is satisfied if
\[
\lim_{k \to \infty} \left[ \delta_{2k} + \max_{H_{2k}, H_{2k+1}} \ell_{\pm}^{(2k+1)}(\sigma) \right] = 0 \quad (6.15)
\]
and, in Assumption 5.1,
\[
\delta(\sigma) = \begin{cases} 
1/2 & \left| \frac{d}{d\sigma} \ell_{+}^{(2k+1)}(\sigma) \right| + \left| \frac{d}{d\sigma} \ell_{-}^{(2k+1)}(\sigma) \right| \\
0 & \text{in } R,
\end{cases}
\] in \( P \).

Hence if (6.15) and
\[
\lim_{k \to \infty} \max_{H_{2k}, H_{2k+1}} \left| \frac{d}{dx_1} \ell_{\pm}^{(2k+1)}(x_1) \right| = 0 \quad (6.16)
\]
are satisfied, then
\[ \sigma_e(-\Delta_{\Omega}) = \sigma_e(H_\Gamma). \]
The operator \( H_\Gamma \) is defined by
\[ H_\Gamma F = -\frac{1}{\alpha} (\alpha F')' \]
and has domain
\[ \mathcal{D}(H_\Gamma) = \{ F : F, \alpha F' \in AC_{\text{loc}}[0, b], \ F, H_\Gamma F \in L^2(\Gamma), \ F'(0) = 0 \} \] (6.17)
since the conditions in Theorem 3.3 at the vertices \( H_k, k \geq 1, \) require only the continuity of \( F' \) in view of (6.14).

It is easy to show that \( \sigma_e(H_\Gamma) = [0, \infty) \) if
\[
\lim_{k \to \infty} h_{2k} = \infty. \quad (6.18)
\]

For let \( \lambda > 0 \) and
\[
F_k(t) = \begin{cases} 
ck G_k(\sigma)e_k(\sigma), & 0 \leq \sigma \leq h_{2k}, \\
0, & \text{otherwise,}
\end{cases}
\]
where \( c_k \) is constant to make \( \| F_k \|_\Gamma = 1 \),

\[
\begin{align*}
G_k(\sigma) &= \left\{ \sigma(\sigma - h_{2k}) \right\}^2, \\
e_k(\sigma) &= e^{\sqrt{\sigma}},
\end{align*}
\]

and \( k \to \infty \) along a sequence for which \( h_{2k} \to \infty \). Then \( F_k \in C^2(H_{2k}, H_{2k+1}) \cap C^1(\Gamma) \), and a straightforward calculation gives

\[
\| F_k \|_\Gamma^2 \asymp |c_k|^2 \delta_{2k} h_{2k}^9,
\]

and, with \( |c_k| \asymp \delta_{2k}^{1/2} h_{2k}^{-9/2} \),

\[
\|(H_{\Gamma} - \lambda) F_k\|_\Gamma \asymp O(h_{2k}^{-1}),
\]

where \( \asymp \) means that the ratio of the two sides is bounded above and below by positive constants which are independent of \( k \). It follows that \( \{ F_k \} \) is a singular sequence for \( H_{\Gamma} \) at \( \lambda \) and the assertion is established.

References