Some remarkable sine and cosine functions

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ABSTRACT. – A generalization of the familiar trigonometric and hyperbolic functions is described. These functions are the inverses of some very special Abelian integrals. Many well-known formulas, for example, \( \sin^2 x + \cos^2 x = 1 \), \( \tan x \cot x = 1 \), and \( \cosh z = \cos(iz) \), have their counterpart.

KEY WORDS: Abelian integrals, special functions, non-linear eigenvalue problems.

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1. Introduction

While studying a non-linear eigenvalue problem, we were led away by the unexpected appearance of certain special functions akin to the familiar trigonometric functions. They have a sufficiently rich structure, revealed by beautiful formulas, to motivate a study of a topic, perhaps best characterized as nineteenth century mathematics. Though these “new functions” are the inverses of some Abelian integrals, we have not been able to detect them in the classical works. Neither Jacobi nor Abel paid any attention to this peculiar configuration\(^1\).

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\(^1\)(Note added in proof) Jaak Peetre has recently discovered that these functions have been defined and studied by Erik Lundberg in 1879. A copy of Lundberg’s work
The sine function that we have in mind is a solution to a non-linear eigenvalue problem studied\(^2\) by M. Otani in 1984, cf. [Ô]. It was a surprise for us to learn from [P2] that the corresponding hyperbolic functions have been encountered by J. Peetre in 1972 in connection with the determination of a K-functional, cf. [P1]. In [L] we discovered the counterpart to the formula \(\cos^2 x + \sin^2 x = 1\), thus establishing a bridge between the sines and cosines.

The inverse functions of the integrals

\[
(1.1) \quad z = \int_0^w \frac{d\zeta}{(1 - \frac{\zeta}{p-1})^{1/p}} \quad \text{and} \quad z = \int_w^{\sqrt[p]{p-1}} \frac{d\zeta}{(1 - \frac{\zeta}{p-1})^{1/p}} \quad (1 < p < \infty)
\]

are denoted by \(w = w(z) = \sin_p(z)\) and \(w = w(z) = \cos_p(z)\). They are related by the remarkable formula

\[
(1.2) \quad \frac{(\cos_q(z))^q}{q-1} + \frac{(\sin_p(z))^p}{p-1} = 1 \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)
\]

where the exponents \(p\) and \(q\) are conjugated, i.e., \(1/p + 1/q = 1\). Here the reader can recognize the trigonometric formula \(\cos^2 z + \sin^2 z = 1\), to which the above reduces for \(p = q = 2\). The derivation formulas become

\[
(1.3) \quad \frac{d\sin_p(z)}{dz} = (p - 1)^{1/p}(\cos_q(z))^{q-1},
\]

\[
\frac{d\cos_p(z)}{dz} = -(p - 1)^{1/p}(\sin_q(z))^{q-1}.
\]

To pursue the analogy with trigonometric functions further, we mention that the first positive zero of \(\sin_p(x)\) is

\[
(1.4) \quad \pi_p = \frac{2\sqrt[p]{p-1}}{p \sin \frac{\pi}{p}} \pi
\]
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so that $\pi_2 = \pi$. Note that the important formula

\[
\pi_p = \pi_q, \quad \frac{1}{p} + \frac{1}{q} = 1
\]

holds. Indeed, $2\pi_p$ is the common period of $\sin_p(x), \sin_q(x), \cos_p(x), \cos_q(x)$. Moreover, $\sin_p(x) = \cos_p\left(\frac{\pi_p}{2} - x\right)$ and $\cos_p(x) = \sin_p\left(\frac{\pi_p}{2} - x\right)$, as expected.

The inverse function to

\[
z = \int_0^w \frac{d\zeta}{1 + \frac{\zeta^p}{p-1}} \quad (1 < p < \infty)
\]

is denoted by $w = w(z) = \tan_p(z)$. As we shall see,

\[
\tan_p(z) = \frac{\sin_p(z)}{\sin'_p(z)} = \frac{\sin_p(z)}{(p-1)^{1/p}(\cos_q(z))^{q-1}}
\]

where $1/p + 1/q = 1$. We also have

\[
\frac{d\tan_p(z)}{dz} = 1 + \frac{(\tan_p(z))^p}{p-1}.
\]

The corresponding hyperbolic functions were encountered by J. Peetre. In [P1] he defined the functions $C_p(x)$ and $S_p(x)$ as inverses of the integrals

\[
x = \int_1^w \frac{dt}{(tp - 1)^{1/p}} \quad \text{and} \quad x = \int_0^w \frac{dt}{(1 + tp)^{1/p}}.
\]

In this case

\[
C^q_p\left(\frac{x}{\sqrt[q-1]{q}}\right) - S^p_p\left(\frac{x}{\sqrt[p-1]{p}}\right) = 1 \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).
\]

We prefer a more consistent normalization\(^3\), thus dividing $t^p$ by $p - 1$ in the integrands and replacing the lower limit 1 in the first integral by $\sqrt[p]{p - 1}$. Then we have

\[
\frac{1}{q - 1} - \frac{(sh_p(z))^p}{p - 1} = 1 \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).
\]

\(^3\)The ubiquitous roots $\sqrt[p]{p - 1}, \sqrt[q]{q - 1}$ and factors like $p - 1, q - 1$, though annoying as they are, cannot be eliminated by pure normalization. They will always enter into most of the formulas.
in easily understandable notation. The reader will recognize the formula
\( \text{ch}^2 z - \text{sh}^2 z = 1 \). (Formulas (1.9) and (1.10) are new).

The important formulas \( \text{ch} z = \cos(iz) \) and \( \text{sh} z = -i \sin(iz) \) have
t heir counterparts. This is the reason for why we insist on defining the
functions for complex variables. The power series expansions in Section 6
will exhibit this connexion more clearly.

What has been written down so far is neither rigorous, nor quite cor-
rect. The \( p \)th root is not single valued and at least one of the quantities
\( a^p, a^q, a^{p-1}, a^{q-1}, \ldots \) is ambiguous. And what is more important is that
the defining integrals are "many-valued functions" of the upper limit \( w \) so
that Riemann surfaces must be introduced in a proper way.

It is clear that the rigorous standard methods in Complex Analysis
needed, are not so interesting in this peculiar setting. Once discovered, our
formulas are usually easy to verify. Therefore we are interested only in the
purely formal aspects and so we have chosen a descriptive approach. Often
some additional arguments are needed. Section 7 is incomplete.

Needless to say, there is always a drawback in any genuine generali-
ization. In our case it is that we have not been able to find any useful
addition formulas. Indeed, a theorem of Weierstrass seems to indicate that
there are no such formulas\(^4\). The addition formulas for Abelian integrals
seem to be close to the best that one can hope for, but they do not work
for the inverse functions. To take an example, for \( p = 4 \), we can express
the integral defining \( \tan_4(z) \) in terms of logarithms. The resulting addition
formula for \( \arctan_4(w) \) is given in Section 8.

In conclusion, the elementary functions \( \sin z, \cos z, \tan z, \cot z, \sh z, \)
\( \text{ch} z, \text{th} z, \text{cth} z \) and their inverse functions have a fascinating generali-
ation. The main features of this theory will be given here, although only at
a formal level.

2. A differential equation

The function \( \sin_p(x) \) is the solution to a nonlinear eigenvalue problem.
This is equivalent to a minimization problem in the Calculus of Variations.

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\(^4\)However, see Lecture 4 (Addition Formulas) in R. Askey: Orthogonal Polynomials
and Special Functions, SIAM, Bristol 1975.
The minimum $\lambda_p$ of the Rayleigh quotient

$$R_p(u) = \frac{\int_{a}^{b} |u'(x)|^p dx}{\int_{a}^{b} |u(x)|^p dx} \quad (1 < p < \infty)$$

taken among all real-valued functions $u \in C^1[a,b]$ with $u(a) = u(b) = 0$ is equal to the first eigenvalue $\lambda$ of the equation

$$\frac{d}{dx}(|u'|^{p-2}u') + \lambda |u|^{p-2}u = 0.$$ 

The resulting sharp estimate $\sqrt[p]{\lambda_p} ||u||_p \leq ||u'||_p$ is called Wirtinger’s inequality in the classical case $p = 2$, when the equation reduces to $u'' + \lambda u = 0$. The eigenvalues and the eigenfunctions have been thoroughly studied by M. Ôtani. The eigenvalues are $\lambda, 2\lambda, 3\lambda, \ldots$ where

$$\lambda_p = (p-1) \left\{ \frac{2}{b-a} \int_{0}^{1} \frac{dt}{(1 - t^p)^{1/p}} \right\}^p$$

according to [Ô, p.28]. This follows from the "first integral"

$$\frac{d}{dx} \left\{ |u'|^p + \frac{\lambda |u|^p}{p-1} \right\} = 0.$$ 

Let us normalize the situation so that $a = 0$ and $b = \pi_p$, where

$$\pi_p = 2 \int_{0}^{\sqrt[p-1]{p-1}} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} = \frac{2 \sqrt[p-1]{p-1}}{p \sin \frac{\pi}{p} \pi}.$$ 

We have $\pi_2 = \pi$. The first (normalized) eigenfunction $u$ is a solution to the equation

$$\frac{\pm du}{dx} = \left( 1 - \frac{u^p}{p-1} \right)^{1/p}.$$
where the plus sign is valid, when \( 0 \leq x \leq \pi_p/2 \), and the minus sign, when \( \pi_p/2 \leq x \leq \pi_p \). Define the solution \( \sin_p(x) \) by

\[
(2.5) \quad x = \int_0^{\sin_p(x)} \frac{dt}{(1 - \frac{t}{p-1})^{1/p}}, \quad 0 \leq x \leq \frac{\pi_p}{2}
\]

and \( \sin_p(\frac{\pi_p}{2} + x) = -\sin_p(\frac{\pi_p}{2} - x) \) for \( 0 \leq x \leq \frac{\pi_p}{2} \). Thus \( \sin_p(x) \) is defined in the interval \([0, \pi_p]\). It is positive, when \( 0 < x < \pi_p \). The maximum value is \( \sin_p(\frac{\pi_p}{2}) = \sqrt{p-1} \). Any first eigenfunction in \([0, \pi_p]\) is of the form \( u = u(x) = C \sin_p(x) \). When \( p \neq 2 \), this sine function is not real analytic in the whole interval. It is real analytic except at the midpoint \( \pi_p/2 \).

According to [Ö, Lemma 3], \( \sin_p(x) \) has a continuous second derivative in \([0, \pi_p]\), if \( 1 < p < 2 \), and a continuous first derivative in \([0, \pi_p]\), if \( 2 < p < \infty \) (in this latter case the continuity of the second derivative breaks down at the midpoint \( \pi_p/2 \)). In any case, \( \sin_p(x) \) is real analytic in the open intervals \([0, \pi_p/2]\) and \([\pi_p/2, \pi_p]\).

Now it is natural to extend the definition of the sine function to the whole real axis as an odd periodic function, i.e.,

\[
(2.6) \quad \sin_p(-x) = -\sin_p(x), \quad \sin_p(x + 2\pi_p) = \sin_p(x).
\]

The period is \( 2\pi_p \). This extension has the advantage that all eigenfunctions can be represented as

\[
\sin_p(x), \sin_p(2x), \sin_p(3x), \ldots
\]

possibly multiplied by normalizing constants\(^4\). The corresponding eigenvalues are \( 1, 2^p, 3^p, \ldots \). All this agrees with [Ö, Theorem 1, Remark 8], as some simple calculations show.

Let us return to the equation (2.2) and write it in the form

\[
(2.7) \quad (p - 1)|u''|^{p-2}u'' + \lambda|u|^{p-2}u = 0.
\]

\(^4\) We cannot resist mentioning that the series

\[
b_1 \sin_p(x) + b_2 \sin_p(2x) + b_3 \sin_p(3x) + \ldots
\]

is akin to the Fourier sine series, but we do not pursue this matter any further.
The choice of the interval $[0, \pi_p]$ in the eigenvalue problem yielded $\lambda_p = 1$. For negative values of $\lambda$ this is not an eigenvalue problem and we cannot insist on having zero end point values. For $\lambda = -1$ we proceed in analogy with the classical case $u'' - u = 0$ and define

\begin{equation}
(2.8) \quad x = \frac{\text{sh}_p(x)}{\int_0^x \frac{dt}{(1 + \frac{tp}{p-1})^{1/p}}} \quad (x \geq 0).
\end{equation}

Here J. Peetre made the choice $\lambda = -(p - 1)$ and ended up with the seemingly simpler definition

\begin{equation}
(2.9) \quad x = \frac{S_p(x)}{\int_0^x \frac{dt}{(1 + t^p)^{1/p}}} \quad (x \geq 0)
\end{equation}

in [P1], but this apparent simplicity is abandoned here, the reason being that, at least formally, $\text{sh}_p(z) = \sqrt{-1} \sin_p(z) / \sqrt{-1}$ in the complex domain. Now the fundamental rule $\pi_p = \pi_q$ for conjugate exponents $p$ and $q$ would be violated if one defined the sine function via $S_p(z)$. Compare formulas (1.9) and (1.10) to see the point.

3. The trigonometric functions

In this section we shall define and study the counterparts to the trigonometric functions, but only for real variables. Observe that

\begin{equation}
(3.1) \quad \frac{\pi_p}{2} = \int_0^\infty \frac{dt}{(1 - \frac{tp}{p-1})^{1/p}} = \int_0^\infty \frac{dt}{1 + \frac{tp}{p-1}} = \frac{\sqrt{p - 1}}{p \sin \frac{\pi}{p}}.
\end{equation}

(The first integral has been evaluated already by Euler). Define

\begin{equation}
(3.2) \quad x = \frac{\sin_p(x)}{\int_0^{\cos_p(x)} \frac{dt}{(1 - \frac{tp}{p-1})^{1/p}}} = \frac{\sqrt{p - 1}}{p \sin \frac{\pi}{p}}.
\end{equation}
for \( 0 \leq x \leq \frac{\pi p}{2} \). We can read off the values

\[
\sin_p \left( \frac{\pi p}{2} \right) = \sqrt[p]{p-1}, \quad \cos_p \left( \frac{\pi p}{2} \right) = 0, \quad \tan_p \left( \frac{\pi p}{2} \right) = \infty,
\]
\[
\cot_p \left( \frac{\pi p}{2} \right) = 0, \quad \sin_p(0) = \tan_p(0) = 0, \quad \cos_p(0) = \sqrt[p]{p-1}.
\]

Extend the definition of the sine to the whole real axis as in Section 2. This definition, done according to the requirements of the eigenvalue problem, leads to the strange formula

\[
(3.3) \quad x = \int_0^{\sin_p(x)} \frac{dt}{(1 - \frac{|t|^p}{p-1})^{1/p}}
\]

for \(-\frac{\pi p}{2} \leq x \leq \frac{\pi p}{2}\). The cosine will be extended to the interval \([-\frac{\pi p}{2}, \pi_p]\) as
\[
\cos_p(\pi_p + x) = -\cos_p\left(\frac{\pi p}{2} - x\right)
\]
and then to \([-\pi_p, 0]\) as an even function, i.e., \(\cos_p(x) = \cos_p(-x)\). Finally, we extend it periodically to the whole real axis: \(\cos_p(x + 2\pi_p) = \cos_p(x)\). In a similar way the tangent is defined as an odd function with the period \(\pi_p\). Strictly speaking, it is not properly defined at the points \(\frac{\pi p}{2} + n\pi_p\). For example, \(\tan_p\left(\frac{\pi p}{2}\right) = \pm \infty\). Also the definition of the cotangent is extended exactly as in the case \(p = 2\).

We have the reduction formulas

\[
\sin_p(x + 2\pi_p) = \sin_p(x), \quad \sin_p(-x) = -\sin_p(x), \quad \sin_p\left(\frac{\pi p}{2} - x\right) = \sin_p\left(\frac{\pi p}{2} + x\right),
\]
\[
\cos_p(x + 2\pi_p) = \cos_p(x), \quad \cos_p(-x) = \cos_p(x), \quad \cos_p\left(\frac{\pi p}{2} - x\right) = -\cos_p\left(\frac{\pi p}{2} + x\right),
\]
\[
\tan_p(x + \pi_p) = \tan_p(x), \quad \tan_p(-x) = -\tan_p(x), \quad \cot_p(-x) = -\cot_p(x).
\]

just to mention a few. Suppose that \(0 \leq x \leq \pi_p/2\) and that \(s = \sin_p(x)\). Then

\[
\frac{\pi p}{2} = \int_0^{\sqrt[p]{p-1}} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} + \int_s^{\sqrt[p]{p-1}} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} + \int_s^0 \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} + \int_0^s \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}},
\]

and moving \(x\) to the left hand side, we see that \(\cos_p\left(\frac{\pi p}{2} - x\right) = \sin_p(x)\). Using the extension formulas we can check that this formula holds for any
real $x$. A similar device works for the other functions as well. Thus we obtain the translation formulas

$$\begin{align*}
\sin_p(x) &= \cos_p \left( \frac{\pi p}{2} - x \right), \\
\cos_p(x) &= \sin_p \left( \frac{\pi p}{2} - x \right)
\end{align*}$$

(3.5)

$$\begin{align*}
\tan_p(x) &= \cot_p \left( \frac{\pi p}{2} - x \right), \\
\cot_p(x) &= \tan_p \left( \frac{\pi p}{2} - x \right).
\end{align*}$$

But the most important relationship between these functions is by far the formula

$$\frac{\left| \sin_p(x) \right|^p}{p-1} + \frac{\left| \cos_q(x) \right|^q}{q-1} = 1 \quad \left( \frac{1}{p} + \frac{1}{q} = 1 \right)$$

(3.6)

with conjugate exponents $p$ and $q$. This phenomenon was essentially detected in [L] with the aid of the observation that $\sqrt[p]{\lambda_p} = \sqrt[q]{\lambda_q}$ for the eigenvalues in (2.3). Once given, formula (3.6) is easily verified. To this end, suppose to begin with that $0 \leq x \leq \pi_p/2$. The substitution

$$\frac{t^p}{p-1} + \frac{\tau^q}{q-1} = 1, \quad qt^{p-1} \, dt + pr^{q-1} \, d\tau = 0$$

yields

$$x = \int_0^s \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} = \int_? \frac{d\tau}{(1 - \frac{\tau^q}{q-1})^{1/q}}$$

where $s = \sin_p(x)$ and the lower limit of integration in the second integral is

$$? = \left\{ (q - 1) \left( 1 - \frac{s^p}{p-1} \right) \right\}^{1/q}.$$

By definition this quantity is $\cos_q(x)$, see (3.2). Simplifying the equation ”$? = \cos_q(x)$” we immediately arrive at (3.6). The result is extended to the whole real axis with the aid of (3.4).

In the same way, but starting with the substitution $t^p = \tau^{-q}$ in the defining integral for $\tan_p(x)$, one arrives at the formula

$$\left| \tan_p(x) \right|^p \left| \cot_q(x) \right|^q = 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$
This is the counterpart to \( \tan(x) \cot(x) = 1 \).

The formulas

\[
\tan_p(x) = \frac{\sin_p(x)}{\sqrt[p]{p-1} |\cos_q(x)|^{q-2} \cos(x)},
\]

\( (3.8) \)

\[
\cot_p(x) = \frac{\cos_p(x)}{\sqrt[p]{p-1} |\sin_q(x)|^{q-2} \sin(x)},
\]

where \( p \) and \( q \) are conjugated, are obtained in a similar way. Substituting

\[
t^p = \frac{\tau_p}{1 + \tau_p}, \quad \tau^p = \frac{t^p}{1 - t^p}
\]
in the integral

\[
x = \int_0^{\sin_p(x)} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}} = \sqrt[p]{p-1} \int_0^{(p-1)^{-1/p} \sin_p(x)} \frac{dt}{(1 - t)^{1/p}}
\]

and doing the usual calculations, we obtain the expression for the tangent in \( (3.8) \).

At this stage we have developed sufficiently many relations between our functions in order to write down their derivation formulas in a succinct form. From \( (3.2) \) we have that

\[
\frac{d \sin_p(x)}{dx} = \left(1 - \frac{(\sin_p(x))^p}{p-1}\right)^{1/p}, \quad \frac{d \cos_p(x)}{dx} = -\left(1 - \frac{(\cos_p(x))^p}{p-1}\right)^{1/p}
\]

\[
\frac{d \tan_p(x)}{dx} = 1 + \frac{(\tan_p(x))^p}{p-1}, \quad \frac{d \cot_p(x)}{dx} = -\left(1 + \frac{(\cot_p(x))^p}{p-1}\right)^{1/p},
\]

when \( 0 \leq x \leq \frac{\pi}{2} \). For other values of \( x \) the reduction formulas \( (3.4) \) must be used. Using \( (3.6) \) or \( (3.8) \) we have

\[
\frac{d \sin_p(x)}{dx} = \sqrt[p]{p-1} |\cos_q(x)|^{q-2} \cos_q(x),
\]

\[
\frac{d \cos_p(x)}{dx} = -\sqrt[p]{p-1} |\sin_q(x)|^{q-2} \sin_q(x),
\]

\( (3.9) \)

\[
\frac{d \tan_p(x)}{dx} = 1 + \frac{|\tan_p(x)|^p}{p-1} = \frac{q-1}{|\cos_q(x)|^q},
\]

\[
\frac{d \cot_p(x)}{dx} = -\left(1 + \frac{|\cot_p(x)|^p}{p-1}\right) = -\frac{q-1}{|\sin_q(x)|^q}
\]
for all real \( x \). It is worth noting that (3.8) can now be written as

\[
\tan_p(x) = \frac{\sin_p(x)}{d \sin_p(x)/dx}, \quad \cot_p(x) = \frac{\cos_p(x)}{d \cos_p(x)/dx}.
\]

(This yields an independent proof of (3.7)).

As an example, let us mention that in the interesting expression

\[
\cot_q(x) - \tan_p(x) = \frac{(p - 1)^{1/p} \cos_q(x)^q - (q - 1)^{1/q} \sin_p(x)^p}{(q - 1)^{1/q}(p - 1)^{1/p} \sin_p(x)^p \cos_q(x)^q - 2 \sin_p(x)^p \cos_q(x)^q}
\]

the derivative of the denominator is equal to \((-pq)\) times the numerator.

4. The hyperbolic functions

The counterparts to the hyperbolic functions will be defined as the inverses of integrals. Define

\[
\begin{align*}
\text{sh}_p(x) &= \int_0^x \frac{dt}{\left(1 + \frac{tp}{p-1}\right)^{1/p}}, \\
\text{ch}_p(x) &= \int \frac{dt}{\sqrt[p-1}{\left(\frac{tp}{p-1} - 1\right)^{1/p}}}
\end{align*}
\]

for \( x \geq 0 \) and use the extensions

\[
\text{sh}_p(-x) = -\text{sh}_p(x), \quad \text{ch}_p(-x) = \text{ch}_p(x)
\]

so that \text{sinus hyperbolicus} is odd and \text{cosinus hyperbolicus} even. Observe that \( \text{sh}_p(0) = 0 \) and \( \text{ch}_p(0) = \sqrt[p-1]{q} \). As in the trigonometric case, a suitable change of variables in the defining integrals leads to the fundamental formula

\[
\frac{\left(\text{ch}_p(x)\right)^p}{p-1} - \frac{\left|\text{sh}_q(x)\right|^q}{q-1} = 1, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

for conjugated exponents \( p \) and \( q \).

Define

\[
\begin{align*}
\text{th}_p(x) &= \int_0^x \frac{dt}{1 - \frac{tp}{p-1}}, \\
\text{cth}_p(x) &= \int \frac{dt}{\frac{tp}{p-1} - 1}
\end{align*}
\]
for \( x \geq 0 \) and use the extension

\[
(4.5) \quad \text{th}_p(-x) = -\text{th}_p(x), \quad \text{cth}_p(-x) = -\text{cth}_p(x)
\]

so that tangens hyperbolica and cotangens hyperbolica are odd functions. Observe, that \(|\text{th}_p(x)| < \sqrt{p - 1}\) while \(|\text{cth}_p(x)| > \sqrt{p - 1}\). We have \(\text{cth}_p(\pm 0) = \pm \infty\). Again we have

\[
(4.6) \quad |\text{th}_p(x)|^p |\text{cth}_q(x)|^q = 1, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

as a simple change of variables in the defining integrals shows.

In a similar fashion one arrives at the relations

\[
\text{th}_p(x) = \frac{\text{sh}_p(x)}{\sqrt{p - 1}(\text{ch}_q(x))^{q-1}},
\]

\[
(4.7) \quad \text{cth}_p(x) = \frac{\text{ch}_p(x)}{\sqrt{p - 1}|\text{sh}_q(x)|^{q-2}\text{sh}_q(x)}.
\]

One cannot help observing that in this setting there is a formal similarity between the hyperbolic and the trigonometric formulas. Indeed, substituting \(\sqrt{-1} t\), where \(\sqrt{-1}\) means \(e^{\pm \pi / p}\), for \( t \) in (4.1) or (4.4) leads us back to the trigonometric integrals (3.2). This suggests that

\[
\text{sh}_p(x) = \sqrt{-1} \sin_p \left( \frac{x}{\sqrt{-1}} \right), \quad \text{ch}_p(x) = \cos_p \left( \frac{x}{\sqrt{-1}} \right),
\]

\[
(4.8) \quad \text{th}_p(x) = \sqrt{-1} \tan_p \left( \frac{x}{\sqrt{-1}} \right), \quad \text{cth}_p(x) = \sqrt{-1} \cot_p \left( \frac{x}{\sqrt{-1}} \right).
\]

Note that \(\sqrt{-1} \sqrt{-1} = -1\). This requires an extension of the definitions to the complex plane. It will be done in Section 7. But let us already at this stage give an obvious explanation for two formulas that Peetre derived through some asymptotic calculations. We have in mind [P2, Section 5, Eqn (6)].

Renaming the variable in

\[
\text{ch}_p \left( \frac{x}{\sqrt{-1}} \right) = \cos_p(x) = \sin_p \left( x + \frac{\pi p}{2} \right) = \sqrt{-1} \text{sh}_p \left( \frac{x}{\sqrt{-1}} + \frac{\pi p}{2\sqrt{-1}} \right)
\]
we have the first of Peetre’s formulas
\[ \operatorname{ch}_p(x) = \sqrt[p]{-1} \operatorname{sh}_p \left( x + \frac{\pi_p}{2 \sqrt[p]{-1}} \right), \]
(4.9)
\[ \operatorname{sh}_p(x) = \sqrt[p]{-1} \operatorname{ch}_p \left( x - \frac{\pi_p \sqrt[p]{-1}}{2} \right). \]

The second follows in the same way. For the sake of completeness we mention the derivation rules

(4.10)
\[ \frac{d \operatorname{sh}_p(x)}{dx} = \sqrt[p]{-1} \operatorname{ch}_q(x)^{q-1}, \quad \frac{d \operatorname{ch}_p(x)}{dx} = \sqrt[p]{-1} \operatorname{sh}_q(x)^{q-2} \operatorname{sh}_q(x) \]
\[ \frac{d \operatorname{th}_p(x)}{dx} = 1 - \frac{|\operatorname{th}_p(x)|^{p}}{p-1}, \quad \frac{d \operatorname{cth}_p(x)}{dx} = 1 - \frac{|\operatorname{cth}_p(x)|^{p}}{p-1}. \]

5. A geometric interpretation

The elementary definition of trigonometric functions as parametrizing
the unit circle \( x^2 + y^2 = 1 \) (and of hyperbolic functions as parametrizing
the hyperbola \( x^2 - y^2 = 1 \)) has a counterpart for general values of \( p \). To
this end, consider the curve

(5.1)
\[ \frac{x^q}{q-1} + \frac{y^p}{p-1} = 1 \quad \left( \frac{1}{p} + \frac{1}{q} = 1, \ x \geq 0, y \geq 0 \right) \]

where we have restricted ourselves to the first quadrant of the plane. The
curve is parametrized by

(5.2)
\[ x = \cos_q(t), \ y = \sin_p(t) \quad \left( 0 \leq t \leq \frac{\pi_p}{2} \right) \]
as Eqn (3.6) shows. The parameter \( t \) has a very simple geometric interpre-
tation. Let \( \text{area}(S_t) \) denote the area of the sector bounded by the x-axis,
the radius from the origin to the point \((\cos_q(t), \sin_p(t))\), and the curve (5.2).
Let \( \text{area}(\Delta_t) \) denote the area of the triangle with corners at the origin, at the point \((\sqrt[p-1]{t},0)\) and \((\cos_q(t), \sin_p(t))\), that is

\[
\text{area}(\Delta_t) = \frac{1}{2} \cos_q(t) \sin_p(t).
\]

The remarkable interpretation of the parameter \( t \) is

\[
t = \frac{p}{(p-1)^{1/p}} \text{ area}(S_t) + \frac{2-p}{(p-1)^{1/p}} \text{ area}(\Delta_t).
\]

This expression is symmetric, that is

\[
t = \frac{q}{(q-1)^{1/q}} \text{ area}(S_t) + \frac{q-2}{(q-1)^{1/q}} \text{ area}(\Delta_t).
\]

If \( p = 2 \), we have the well-known interpretation of \( t \) as two times the area of the circular sector, cf. [M]. To arrive at this, one has just to integrate the equation

\[
1 = \frac{\cos_q(t) \sin'_p(t)}{(q-1)^{1/q}} - \frac{\sin_p(t) \cos'_q(t)}{(p-1)^{1/p}}
\]

from zero to \( t \), see (3.6) and (3.9).

The special case \( t = \frac{\pi}{2} \) yields that the area of the domain in the first quadrant bounded by the coordinate axes and the curve (5.1) is equal to

\[
\frac{(p-1)^{1/p} \pi_p}{2p}.
\]

This area was calculated already by Dirichlet, cf. [Di].

There is a similar interpretation in the hyperbolic case, now based on the curve

\[
\frac{x^q}{q-1} - \frac{y^p}{p-1} = 1.
\]

6. Power series

In [P2] J. Peetre has given extensive power series expansions for his functions \( S_p(x) \) and \( C_p(x) \). One has just to remember that

\[
S_p \left( \frac{x}{\sqrt[p-1]{p}} \right) = \frac{\sin(x)}{p-1}, \quad C_p \left( \frac{x}{\sqrt[p-1]{p}} \right) = \frac{\cos(x)}{\sqrt[p-1]{p}}.
\]
in order to obtain the corresponding expansions for $\text{sh}_p(x)$ and $\text{ch}_p(x)$. Our derivation below is slightly simpler than Peetre’s approach.

We have the expansions:

\begin{equation}
\cos_p(x) = \sqrt{p-1} \left\{ 1 - \frac{x^q}{q} + \frac{x^{2q}}{2q^2(q+1)} - \frac{3q^2 - q^3 - 1}{6q^3(q-1)(q+1)^2(2q+1)} x^{3q} + \ldots \right\},
\end{equation}

\begin{equation}
\sin_p(x) = x \left\{ 1 - \frac{x^p}{p(p-1)(p+1)} + \frac{1 + 2p - p^2}{2p^2(p-1)^2(p+1)(2p+1)} x^{2p} - \frac{1 + 7p + 13p^2 - 2p^3 - 11p^4 + 4p^5}{6p^3(p+1)^2(p-1)^3(1 + 2p)(1 + 3p)} x^{3p} + \ldots \right\},
\end{equation}

\begin{equation}
\tan_p(x) = x \left\{ 1 + \frac{x^p}{(p-1)(p+1)} + \frac{p}{(p-1)^2(p-1)(2p+1)} x^{2p} + \frac{p(4p^2 + p - 1)}{2(p-1)^3(p+1)^2(2p+1)(3p+1)} x^{3p} + \frac{p(18p^4 - 14p^3 - 34p^2 - 7p + 1)}{3(p-1)^4(p+1)^3(2p+1)(3p+1)(4p+1)} x^{4p} + \ldots \right\}.
\end{equation}

Here $p$ and $q$ are conjugate. Observe that $\cos_p(x)$ is a function of $x^q$, i.e., of the $p - 1^{st}$ root of $x^p$. The functions

\[ \frac{\sin_p(x)}{x} \quad \text{and} \quad \frac{\tan_p(x)}{x} \]

depend only on $x^p$. The radius of convergence is $\frac{\pi p}{2}$ for all three series, when $p \neq 2$.

The corresponding hyperbolic formulas are

\begin{equation}
\text{ch}_p(x) = \sqrt{p-1} \left\{ 1 + \frac{x^q}{q} + \frac{x^{2q}}{2q^2(q+1)} + \frac{3q^2 + q^3 - 1}{6q^3(q-1)(q+1)^2(2q+1)} x^{3q} + \ldots \right\},
\end{equation}
(6.6) \[ \text{sh}_p(x) = x \left\{ 1 + \frac{x^p}{p(p-1)(p+1)} + \frac{1 + 2p - p^2}{2p^2(p-1)^2(p+1)(2p+1)} x^{2p} + \frac{1 + 7p + 13p^2 - 2p^3 - 11p^4 + 4p^5}{6p^3(p+1)^2(p-1)^3(1+2p)(1+3p)} x^{3p} + \ldots \right\}, \]

(6.7) \[ \text{th}_p(x) = x \left\{ 1 - \frac{x^p}{(p-1)(p+1)} + \frac{p}{(p-1)^2(p+1)(2p+1)} x^{2p} - \frac{p(4p^2 + p - 1)}{2(p-1)^3(p+1)^2(2p+1)(3p+1)} x^{3p} + \frac{p(18p^4 - 14p^3 - 34p^2 - 7p + 1)}{3(p-1)^4(p+1)^3(2p+1)(3p+1)(4p+1)} x^{4p} - \ldots \right\}. \]

Also here the radius of convergence is \( \frac{\pi}{2} \) for all three series, when \( p \neq 2 \).

Note that Eqn (4.8) connects the series. One has \( \sqrt{-1} \sqrt{-1} = -1 \). The simplest way to obtain these expansions seems to be to start with the tangent. Suppose therefore that

(6.8) \[ \tan_p(x) = x + a_1 x^{p+1} + a_2 x^{2p+1} + a_3 x^{3p+1} x \ldots \]

and insert this expression into the formula

(6.9) \[ \frac{d \tan_p(x)}{dx} = 1 + \frac{(\tan_p(x))^p}{p-1}. \]

This formula also shows that (6.4) contains the right powers. Now we have

(6.10) \[ (\tan_p(x))^p = x^p [1 + \alpha_1 x^p + \alpha_2 x^{2p} + \alpha_3 x^{3p} + \ldots]. \]

Here the Miller formula

\[ \alpha_n = \frac{1}{n} \sum_{k=0}^{n-1} [p(n-k) - k] \alpha_k \alpha_{n-k} \quad (\alpha_0 = 1) \]
simplifies the calculations, cf. [P2]. Successively we get

\[
\begin{align*}
\alpha_0 &= 1 \\
\alpha_1 &= pa_1, \\
\alpha_2 &= pa_2 + \frac{p(p - 1)}{2} a_1^2 \\
\alpha_3 &= pa_3 + p(p - 3)a_1a_2 + \frac{p(p - 1)(p - 2)}{3 \cdot 2} a_1^3 \\
&\vdots
\end{align*}
\]

after some simplification. Now the coefficients \(a_1, a_2, a_3,\) and \(a_4\) are easily calculated from (6.9). This yields (6.4).

Once one of the series is known, the others follow easily. For example, the coefficients in

(6.11) \[\sin_p(x) = x + b_1x^{p+1} + b_2x^{2p+1} + b_3x^{3p+1} + \ldots\]

are calculated from the equation

\[
\tan_p(x) = \frac{\sin_p(x)}{\sin'_p(x)}.
\]

In other words,

\[
x + \frac{1}{(p - 1)(p + 1)} x^{p+1} + \frac{p}{(p - 1)^2(p + 1)(2p + 1)} x^{2p+1} + \ldots
\]

\[
= \frac{x + b_1x^{p+1} + b_2x^{2p+1} + b_3x^{3p+1} + \ldots}{1 + (p + 1)b_1x^p + (2p + 1)b_2x^{2p} + (3p + 1)b_3x^{3p} + \ldots}
\]

so that a multiplication leads to a simple situation where coefficients may be equated.

This brief sketch is sufficient to indicate what the actual calculations are based on.
7. Analytic continuation

This section is merely formal.

The interplay between trigonometric and hyperbolic functions cannot be properly seen without the use of complex variables. There are three ways to extend our functions to the complex plane.

First, one uses the power series in Section 6. For example, define

\[
\sin_p(z) = z \left\{ 1 - \frac{z^p}{p(p-1)(p+1)} + \frac{1 + 2p + p^2}{2p^2(p-1)^2(p+1)(2p+1)} z^{2p} - \ldots \right\}
\]

for \( z = x + iy \). There are two obstacles here. The series converges only for \( |z| < \pi_p/2 \), when \( p \neq 2 \). Hence analytic continuation must be used for values outside the disc of convergence. If \( p \) is not an integer, then \( z^p \) produces a branch point at the origin. This difficulty is always present, since both \( p \) and \( q \) cannot be integers, when they are conjugated. The other disadvantage is that the periods are not easy to detect in the power series.

Second, one uses the defining integrals. For example, define

\[
z = \sin_p(z) \int_0^{\sin_p(z)} \frac{d\zeta}{(1 - \zeta^p/p)^{1/p}}
\]

where the integration in the \( \zeta \)-plane is along any path avoiding the \( p \)th root of \( p - 1 \). It is by no means so clear that this defines anything at all, if the path of integration leaves the open disc \( |w| < \sqrt[p]{p-1} \). The advantage of (7.2) is that the periods reveal themselves. To make this definition precise, one has to introduce a suitable Riemann surface. The difficulties of such an approach are formidable, if \( p \) is irrational. (By Weyl’s lemma the roots of the equation \( \zeta^p = p - 1 \) form a dense subset of the circle \( |\zeta| = \sqrt[p]{p-1} \) in this case!).

Third, one can use the differential equation. For example, \( w = \sin_p(z) \) is defined as the solution to

\[
\frac{d}{dz} \left( [dw/dz]^{p-1} \right) + w^{p-1} = 0 \quad (w(0) = 0, w'(0) = 1).
\]
To say the least, this approach is not easy to fulfil. Here a warning is needed. The approach in Section 2 was based upon the equation

\[(7.4) \quad \frac{d}{dx} \left(|u'(x)|^{p-2}u'(x)\right) + |u(x)|^{p-2}u(x) = 0\]

and done according to the requirements of the eigenvalue problem. Because of the absolute values present, it seems to us as if (7.3) and (7.4) could produce different solutions.

In conclusion, we want to emphasize that the formulas

\[(7.5) \quad \text{ch}_p(z) = \cos_p\left(\frac{z}{\sqrt{-1}}\right), \quad \text{sh}_p(z) = \sqrt{-1} \sin_p\left(\frac{z}{\sqrt{-1}}\right)\]

always yield the right results.\(^5\) (Here \(\sqrt{-1}\) denotes any primitive root, for example \(e^{i\pi/n}\)). The formulas in Section 4 can easily be obtained from those in Section 3 in this way, and vice versa.

8. On addition formulas

The addition formulas for the ordinary trigonometric and hyperbolic functions are a direct consequence of the fundamental property \(e^{z_1+z_2} = e^{z_1}e^{z_2}\) of the exponential function. For elliptic functions the situation is more involved. We have not been able to find any useful addition formulas at all for our functions. To see the main difficulty, let us compare the two tangents \(\text{th}(z)\) and \(\text{th}_4(z)\). The formula

\[(8.1) \quad \int_0^{w_1} \frac{d\zeta}{1-\zeta^2} + \int_0^{w_2} \frac{d\zeta}{1-\zeta^2} = \int_0^{\frac{w_1+w_2}{1+w_1w_2}} \frac{d\zeta}{1-\zeta^2}\]

is merely the addition formula \(\text{th}(a+b) = \frac{\text{th}_4(a) + \text{th}_4(b)}{1 + \text{th}_4(a) \text{th}_4(b)}\). To simplify the notation, observe that,

\[(8.2) \quad z = \int_0^{w} \frac{d\zeta}{1-\zeta^4}, \text{ if } w = 1 \sqrt[4]{3} \text{th}_4\left(\frac{z}{\sqrt[4]{3}}\right),\]

\(^5\)There is something strange for odd integers \(p\), for example \(\text{ch}_3(x) = \cos_3(-x)\) and \(\text{sh}_3(x) = -\sin_3(-x)\). This is according to the power series, and cannot be ignored.
as a simple change of variables shows.

We have

\[
\frac{4}{1 - \zeta^4} = \frac{1}{1 - \zeta} + \frac{1}{1 + \zeta} + \frac{1}{1 + i\zeta} + \frac{1}{1 - i\zeta},
\]

and integrating we obtain

\[
(8.3) \quad 4 \int_0^w \frac{d\zeta}{1 - \zeta^4} = \ln \left( \frac{1 + w}{1 - w} \right) + i \ln \left( \frac{1 - iw}{1 + iw} \right).
\]

Trying to add two such integrals we have to deal with the situation

\[
\frac{1 + w_1}{1 - w_1} \cdot \frac{1 + w_2}{1 - w_2} = \frac{1 + \frac{w_1 + w_2}{1 + w_1 w_2}}{1 - \frac{w_1 + w_2}{1 + w_1 w_2}}, \quad \frac{1 - iw_1}{1 + iw_1} \cdot \frac{1 - iw_2}{1 + iw_2} = \frac{1 - i \frac{w_1 + w_2}{1 - w_1 w_2}}{1 + i \frac{w_1 + w_2}{1 - w_1 w_2}}.
\]

Unfortunately

\[
\frac{w_1 + w_2}{1 + w_1 w_2} \neq \frac{w_1 + w_2}{1 - w_1 w_2},
\]

except in trivial cases. This will destroy any algebraic formula of the type (8.1) for the integral (8.3).

However, adding several integrals we have, for example

\[
(8.4) \quad \int_0^{w_1} \frac{d\zeta}{1 - \zeta^4} + \int_0^{w_2} \frac{d\zeta}{1 - \zeta^4} + \int_0^{w_3} \frac{d\zeta}{1 - \zeta^4} = \int_0^a \frac{d\zeta}{1 - \zeta^4} + \int_0^b \frac{d\zeta}{1 - \zeta^4}
\]

where \(a\) and \(b\) are algebraic functions of \(w_1, w_2,\) and \(w_3\). Indeed, a direct computation gives

\[
(8.5) \quad a, b = \frac{AB \pm \sqrt{1 + (A^2 - 1)(B^2 + 1)}}{A + B}
\]

where

\[
(8.6) \quad A = \frac{w_1 + w_2 + w_3 + w_1 w_2 w_3}{1 + w_1 w_2 + w_1 w_3 + w_2 w_3}, \quad B = \frac{w_1 + w_2 + w_3 - w_1 w_2 w_3}{1 - w_1 w_2 - w_1 w_3 - w_2 w_3}.
\]

It is worth noting that

\[
(1 - A^2)(1 + B^2) = \frac{(1 - w_1^4)(1 - w_2^4)(1 - w_3^4)}{(1 - s_2^2)^2}, \quad s_2 = w_1 w_2 + w_1 w_3 + w_2 w_3.
\]
Unfortunately, it is not possible to translate the information in (8.4) to the inverse function, that is, no addition theorem for \( \text{th}_4(z) \) can result, except for special exceptional values of \( w_1, w_2, \) and \( w_3 \). Eqn (8.4) is an Abelian addition theorem for the integrals. See [J] for similar results. We have expressed the sum of three integrals as the sum of two. Fewer than five integrals will not do here. But this was the case \( p = 4 \). For example, the case \( p = 8 \) will require nine integrals instead of five. Therefore one cannot hope for anything resonable, when \( p \) is arbitrary.

To derive an Abelian addition theorem for the sum

\[
\int_0^{w_1} \frac{d\zeta}{(1 + \zeta^4)^{1/4}} + \int_0^{w_2} \frac{d\zeta}{(1 + \zeta^4)^{1/4}} + \int_0^{w_3} \frac{d\zeta}{(1 + \zeta^4)^{1/4}} ,
\]

one has just to remember that

\[
\int_0^w \frac{d\zeta}{(1 + \zeta^4)^{1/4}} = \int_0^{w(1+w^4)^{-1/4}} \frac{d\zeta}{1 + \zeta^4}
\]

by the substitution encountered in Section 3. Now (8.4) provides the desired formula.

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