Mean dimension and widths of classes of functions on the line. (Russian)


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Let $1 \leq p \leq \infty$, $r \in \mathbb{N}$. The author denotes by $W_p(R)$ the set of functions $x(\cdot) \in L_p(R)$ such that $x^{(r-1)}(\cdot)$ is a locally absolutely continuous function and $x^{(r)}(\cdot) \in L_p(R)$. $W_p(R)$ is a Banach space with the norm $\|x(\cdot)\|_{L_p(R)} + \|x^{(r)}(\cdot)\|_{L_p(R)}$. Let $W_p^r(R) := \{x(\cdot) \in W_p(R) : \|x^{(r)}(\cdot)\|_{L_p(R)} \leq 1\}$.

The author introduces the mean dimensions of Kolmogorov, Bernstein and linear types, and denotes them by $\bar{d}$, $\bar{b}$, $\bar{\lambda}$, respectively. The main result is as follows. Theorem 1: If $\lambda > 0$, then

$$\bar{b}_N(W_p^r(R), L_p(R)) = \bar{d}_N(W_p^r(R)),$$

$$L_p(R) = \bar{\lambda}_N(W_p^r(R), L_p(R)) = A(p, r) N^{-r},$$

where $A(p, r) = 2^{-r} \|\hat{x}(\cdot)\|_{L_p([0,1])}$ and $x(\cdot)$ is the unique solution for the following extremal problem:

$$\|x(\cdot)\|_{L_p([0,1])} \rightarrow \sup \|x^{(r)}(\cdot)\|_{L_p([0,1])} \leq 1,$$

$$x^{(i)}((1 - (-1)^i)/2) = 0, 0 \leq i \leq r - 1.$$
MEAN DIMENSION, WIDTHS, AND OPTIMAL RECOVERY OF SOBOLEV CLASSES OF FUNCTIONS ON THE LINE

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ABSTRACT. The concept of mean dimension is introduced for a broad class of subspaces of $L_p[\mathbb{R}]$, and analogues of the Kolmogorov widths, Bernstein widths, Gel’fand widths, and linear widths are defined. The precise values of these quantities are computed for Sobolev classes of functions on $\mathbb{R}$ in compatible metrics, and the corresponding extremal spaces and operators are described. A closely related problem of optimal recovery of functions in Sobolev classes is also studied.

INTRODUCTION

The foundation for investigations connected with approximation of functions on $\mathbb{R}$ was laid in the work of S. N. Bernstein. The space of entire functions of exponential type served there as a means of approximation. Splines have been used more and more often in recent decades. In connection with approximation on $\mathbb{R}$, entire functions and splines are infinite-dimensional constructions, and the quantities characterizing the corresponding approximations are expressed in terms reflecting the intrinsic structure of the approximation apparatus (the degree of an entire function, the denseness of the distribution of nodes of a spline, and so on). How do such methods of approximation compare with each other?

One of the possible approaches involves the concept of mean dimension. The question of studying averaged characteristics of classes of (random) functions was first posed by Shannon and had to do with “entropy” averaging. A corresponding deterministic variant was introduced by Kolmogorov. Tikhomirov [1] proposed an analogous characteristic—the mean dimension—for subspaces of functions on the line, starting out not from entropy but from the Kolmogorov width; this characteristic was studied by Đinh Dung and Magaril-Il’yaev [2], by Đinh Dung [3], and by Le Chyong Tung [4].

In this article we introduce for a broad class of spaces including splines and entire functions the concepts of $\varphi$-mean dimension (which is a generalization of the definition in [1]) and codimension, and we define analogues of the Kolmogorov widths, Bernstein widths, Gel’fand widths, and linear widths. The main results are conditions for finiteness of the $\varphi$-mean Kolmogorov widths, precise computations of all these widths for Sobolev classes of functions on the line in compatible metrics, and a description of the corresponding extremal spaces and operators.

The article consists of seven sections (aside from §0, where some definitions and notation of general character are given). The first three are devoted to the concepts of $\varphi$-mean dimension and codimension and to the definition of the $\varphi$-mean widths. In §4 we formulate the main results. §5 contains the assertions which are...
the essential bases for the proofs of the main theorems. Here we present sharp inequalities of Bernstein and Bohr-Favard type, sharp estimates for approximation of functions in the Sobolev classes on the line by interpolation splines, an analogue of the Tikhomirov theorem on the width of a ball for concrete spaces, and so on. In §6 we prove the main results. Finally, in §7 we give a precise solution for a certain problem involving optimal recovery of functions in Sobolev classes on the line, a problem that is closely related to the quantities introduced above and is a natural generalization of the well-known problem of recovering smooth functions from their values at a countable number of points.

The main assertions in this article were announced in the notes [5] and [6]. The author thanks A. P. Buslaev and V. M. Tikhomirov for useful discussions.

§0

Here we present some definitions and notation of general character. Let \( \mathbb{R}, \mathbb{Z}, \mathbb{Z}_+, \) and \( \mathbb{N} \) be the sets of all real numbers, integers, nonnegative integers, and positive integers, respectively.

If \( (X, \| \cdot \|) \) is a normed space (NS), then we use the following notation:

\[
BX := \{ x \in X | \| x \| \leq 1 \}
\]

is the unit ball in \( X \).

\( \text{Lin}(X) \) is the collection of all subspaces of \( X \).

Let \( A, C \subset X \) and \( x \in X \). Then we make the following definitions:

\[
d(x, A, X) := \inf \{ \| x - y \| | y \in A \}
\]

is the distance from \( x \) to \( A \) in \( X \).

\[
d(C, A, X) := \sup \{ d(x, A, X) | x \in C \}
\]

is the best approximation of \( C \) by the set \( A \) in \( X \) (or the deviation of \( C \) from \( A \) in \( X \)).

If \( Y \) is another NS, then \( \mathcal{L}(Y, X) \) is the collection of all continuous linear operators from \( Y \) to \( X \).

The notation \( Y \hookrightarrow X \) means that the NS \( Y \) is continuously imbedded in the NS \( X \), i.e., \( Y \subset X \) and the imbedding operator is continuous.

If \( A \in \mathcal{L}(Y, X) \), then \( \ker A \) and \( \text{Im} A \) (or \( A(Y) \)) denote the kernel and range of the operator \( A \), respectively.

Let \( C \) be a centrally symmetric subset of \( X \), and let \( n \in \mathbb{Z}_+ \). We recall that the Kolmogorov \( n \)-width of the subset \( C \) of \( X \) is defined to be

\[
d_n(C, X) := \inf \{ d(C, L, X) | L \in \text{Lin}(X) \}
\]

where the infimum is taken over all \( L \in \text{Lin}(X) \) such that \( \dim L \leq n \), and the Bernstein \( n \)-width of a subset \( C \) of \( X \) is defined to be

\[
b_n(C, X) := \sup \{ d(C, L, X) | L \in \text{Lin}(X) \}
\]

where the supremum is taken over all \( L \in \text{Lin}(X) \) such that \( \dim L \geq n + 1 \).

Let \( I \) be a closed interval of the line \( \mathbb{R} \), and let \( 1 \leq p \leq \infty \). Denote by \( L_p(I) \) the space of \( p \)-integrable (essentially bounded for \( p = \infty \)) real-valued functions on \( I \) with the usual norm.

§1. Definitions of the \( \varphi \)-mean dimension and codimension

1.1. We first give the definition of the \( \varphi \)-mean dimension for a subspace of \( L_p(\mathbb{R}) \).

Suppose that \( 1 \leq p \leq \infty \), \( \alpha > 0 \), and \( x(\cdot) \in L_p(\mathbb{R}) \). Define \( P_{\alpha}x(t) = x(t) \) if \( t \in [-\alpha, \alpha] \) and \( P_{\alpha}x(t) = 0 \) if \( t \notin [-\alpha, \alpha] \). It is clear that \( P_{\alpha} \in \mathcal{L}(L_p(\mathbb{R}), L_p(\mathbb{R})) \), \( \| P_{\alpha} \| = 1 \), and \( P_{\alpha} \circ P_{\alpha'} = P_{\alpha} \) if \( \alpha' \geq \alpha \).

Denote by \( \text{Lin}_c(L_p(\mathbb{R})) \) the collection of \( L \in \text{Lin}(L_p(\mathbb{R})) \) such that the restriction of \( P_{\alpha} \) to \( L \) is a compact operator for each \( \alpha > 0 \).
If \( L \in \text{Lin}_c(L_p(\mathbb{R})) \) and \( \alpha > 0 \), then by definition the set \( P_\alpha(L \cap BL_p(\mathbb{R})) \) is precompact, and hence \( d_n(P_\alpha(L \cap BL_p(\mathbb{R})), L_p(\mathbb{R})) \to 0 \) as \( n \to \infty \). Consequently, the quantity
\[
K_\varepsilon(\alpha, L, L_p(\mathbb{R})) := \min \{ n \in \mathbb{Z}_+ | d_n(P_\alpha(L \cap BL_p(\mathbb{R})), L_p(\mathbb{R})) < \varepsilon \},
\]
called the \( \varepsilon \)-dimension of the set \( P_\alpha(L \cap BL_p(\mathbb{R})) \), is finite for all \( \alpha > 0 \) and \( \varepsilon > 0 \).

It is easy to see that the function \( \varepsilon \to K_\varepsilon(\alpha, L, L_p(\mathbb{R})) \) is nonincreasing. We show that the function \( \alpha \to K_\varepsilon(\alpha, L, L_p(\mathbb{R})) \) is nondecreasing. Indeed, suppose that \( \alpha \leq \alpha' \) and \( M' \in \text{Lin}(L_p(\mathbb{R})) \) is such that \( \dim M' = K_\varepsilon(\alpha', L, L_p(\mathbb{R})) \) and \( \dim(P_\alpha(L \cap BL_p(\mathbb{R})), M', L_p(\mathbb{R})) < \varepsilon \). Let \( M = P_\alpha(M') \). It is clear that \( \dim M \leq \dim M' \). For all \( x(\cdot) \in L \cap BL_p(\mathbb{R}) \) and \( y(\cdot) \in M' \) we have (using the properties of \( P_\alpha \)) that
\[
d(P_\alpha(x(\cdot), M, L_p(\mathbb{R})) \leq \|P_\alpha x(\cdot) - P_\alpha y(\cdot)\|_{L_p(\mathbb{R})} \leq \|P_\alpha x(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})}.
\]
Consequently,
\[
d(P_\alpha(x(\cdot), M, L_p(\mathbb{R})) \leq d(P_\alpha(x(\cdot), M', L_p(\mathbb{R})), M', L_p(\mathbb{R})),
\]
and hence \( K_\varepsilon(\alpha, \ldots) \leq K_\varepsilon(\alpha', \ldots) \).

Denote by \( \Phi \) the set of all nondecreasing positive functions \( \varphi(\cdot) \) on \((0, \infty)\) such that \( \varphi(\alpha) \to \infty \) as \( \alpha \to \infty \).

**Definition.** Suppose that \( 1 \leq p \leq \infty, L \in \text{Lin}_c(L_p(\mathbb{R})) \), and \( \varphi(\cdot) \in \Phi \). The quantity
\[
\dim(L, L_p(\mathbb{R}), \varphi(\cdot)) := \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{K_\varepsilon(\alpha, L, L_p(\mathbb{R}))}{\varphi(\alpha)}
\]
is called the \( \varphi \)-mean dimension of \( L \) in \( L_p(\mathbb{R}) \).

If \( \varphi(\alpha) = 2\alpha \), then we call this quantity the mean dimension of \( L \) in \( L_p(\mathbb{R}) \) and denote it by \( \dim(L, L_p(\mathbb{R})) \).

In the last case this is an insignificant modification of the concept of mean dimension proposed by Tikhomirov [1].

1.2. The scheme in §1.1 is easily generalized to the abstract situation. Let \( X \) be an NS, \( \mathfrak{A} \) some ordered set, and \( \mathcal{P} = \{ P_\alpha \}_{\alpha \in \mathfrak{A}} \) a family of operators on \( X \) such that (a) \( \|P_\alpha\| = 1 \) for all \( \alpha \in \mathfrak{A} \), and (b) \( P_{\alpha'} \circ P_\alpha = P_\alpha \) if \( \alpha' \geq \alpha \). Then in a way completely analogous to the preceding we can define (relative to the family \( \mathcal{P} \)) \( \text{Lin}_c(X), K_\varepsilon(\alpha, L, X), \) and \( \dim(L, X, \varphi(\cdot)) \).

Note that if \( \mathcal{P} = \{ I \} \), where \( I \) is the identity operator, then \( \text{Lin}_c(X) \) is the collection of all finite-dimensional subspaces of \( X \), and the \( \varepsilon \)-dimension of a subspace coincides with its usual dimension for \( 0 < \varepsilon < 1 \).

We present an example of another family of operators on \( L_p(\mathbb{R}) \) satisfying the conditions (a) and (b). Let \( \mathfrak{A} = \mathbb{Z}_+ \), \( \alpha \in \mathbb{Z}_+ \), and \( x(\cdot) \in L_p(\mathbb{R}) \). Then \( P_\alpha x(\cdot) \) is defined to be the piecewise constant function equal to \( 2^\alpha \int_{j2^{-\alpha}}^{(j+1)2^{-\alpha}} x(\tau) d\tau \) on the interval \((j2^{-\alpha}, (j+1)2^{-\alpha})\), \( j \in \mathbb{Z} \). It is easy to show that (a) and (b) hold.

1.3. We define the \( \varphi \)-mean codimension of a subspace. Let \( Y \) be an NS of functions on \( \mathbb{R} \) and \( Y^* \) its dual subspace. For each \( \alpha > 0 \) let
\[
Y_\alpha := \{ y(\cdot) \in Y | \text{supp} y(\cdot) \subset [-\alpha, \alpha] \},
\]
and let
\[
Y_\alpha^* := \{ y^* \in Y^* | \langle y^*, y(\cdot) \rangle = 0 \ \forall y(\cdot) \in Y_\alpha \}
\]
be the annihilator of \( Y_\alpha \).
Denote by \( \text{Lin}_r(Y) \) the collection of \( L \in \text{Lin}(Y) \) such that
\[
L := L(A) := \{y(\cdot) \in Y \mid (y^*, y(\cdot)) = 0 \ \forall y^* \in A\}
\]
for some \( A \subset Y^* \), and let
\[
M(\alpha, L(A), Y) := \text{card}\{y^* \in A \mid y^* \notin Y^\perp_{\alpha}\} < \infty
\]
for all \( \alpha > 0 \).
It follows from the last condition that the codimension of \( L \cap Y_\alpha \) in \( Y_\alpha \) is finite.
It is easy to see that the function \( \alpha \to M(\alpha, L(A), Y) \) is nondecreasing.

**Definition.** Suppose that \( Y \) is an NS of functions on \( \mathbb{R} \), \( L(A) \in \text{Lin}_r(Y) \), and \( \varphi(\cdot) \in \Phi \). The quantity
\[
\text{codim}(L(A), Y, \varphi(\cdot)) := \liminf_{\alpha \to \infty} \frac{M(\alpha, L(A), Y)}{\varphi(\alpha)}
\]
is called the \( \varphi \)-mean codimension of \( L(A) \) in \( Y \).
If \( \varphi(\alpha) = 2\alpha \), then we speak of the mean codimension of \( L(A) \) in \( Y \) and denote it by \( \text{codim}(L(A), Y) \).

It is obvious how to carry over this scheme to an arbitrary NS \( Y \) if some family \( \{\{Y_\alpha\}_{\alpha \in \mathbb{A}} \} \) of subspaces is singled out in the latter, where \( \mathbb{A} \) is an ordered set and \( Y_\alpha \subset Y_{\alpha'} \) if \( \alpha \leq \alpha' \).

§2. EXAMPLES OF SPACES OF FINITE \( \varphi \)-MEAN DIMENSION AND CODIMENSION

2.1. Let \( \Xi \) be the collection of sequences \( \xi := \{t_j\}_{j \in \mathbb{Z}} \) of points in \( \mathbb{R} \) such that \( t_j < t_{j+1}, j \in \mathbb{Z} \), and \( |t_j| \to \infty \) as \( j \to \infty \). If \( \xi \in \Xi \) and \( m \in \mathbb{Z}_+ \), then by \( S^m_\xi(R) \) we denote the space of \( m \)th-order polynomial splines of defect 1 on \( \mathbb{R} \) with nodes at the points \( t_j, j \in \mathbb{Z} \). In the case when \( t_j = \theta + jh, j \in \mathbb{Z}, \) for some \( h > 0 \) and \( \theta \in [0, h) \) the corresponding space of splines is denoted by \( S^m_{\theta, h}(R) \), or simply \( S^m_h(R) \) if \( \theta = 0 \).

Let \( \xi \in \Xi \) and \( \alpha > 0 \). The number of points of the sequence \( \xi \) on \((-\alpha, \alpha)\) will be denoted by \( N(\alpha, \xi) \) \( N(\alpha, \xi) = 0 \) if there are no points of \( \xi \) on \((-\alpha, \alpha)\).

Since it is obvious that \( P_\alpha(S^m_\xi(R) \cap B_{L_p}(R)) \) is precompact for any \( \alpha > 0 \) (as a bounded subset of a finite-dimensional space), it follows that \( S^m_\xi(R) \cap L_p(R) \in \text{Lin}_r(L_p(R)) \).

Denote by \( E_\sigma(R) \), where \( \sigma > 0 \), the restriction to \( \mathbb{R} \) of the space of entire functions of exponential type \( \sigma \).

It follows from the Bernstein-Nikol'skii theorem ([7], Theorem 3.3.6) that for all \( 1 \leq p \leq \infty \) and \( \alpha > 0 \) the set \( P_\alpha(E_\sigma(R) \cap B_{L_p}(R)) \) is precompact, and hence \( E_\sigma(R) \cap L_p(R) \in \text{Lin}_r(L_p(R)) \).

**Lemma 2.1.** Suppose that \( 1 \leq p \leq \infty, m \in \mathbb{Z}_+, \xi \in \Xi, \varphi(\cdot) \in \Phi \), and \( \sigma > 0 \). Then:
(a) \[
\dim(S^m_\xi(R) \cap L_p(R), L_p(R), \varphi(\cdot)) = \liminf_{\alpha \to \infty} \frac{N(\alpha, \xi)}{\varphi(\alpha)}.
\]
In particular, if \( h > 0 \) and \( \theta \in [0, h) \), then
\[
\overline{\dim}(S^m_{\theta, h}(R) \cap L_p(R), L_p(R)) = h^{-1}.
\]
(b) \[
\overline{\dim}(E_\sigma(R) \cap L_p(R), L_p(R)) = \sigma/\pi.
\]
Part (b) was actually proved by Tikhomirov for $p = \infty$ [8]. From [3] for $1 < p < \infty$ and from [4] for $p = \infty$ we have the more general fact

$$\dim(B_{G,p}(\mathbb{R}), L_p(\mathbb{R})) = \text{meas } G/2\pi,$$

where $G$ is a bounded Jordan-measurable set, $\text{meas } G$ is its Lebesgue measure, and $B_{G,p}(\mathbb{R})$ is the collection of functions in $L_p(\mathbb{R})$ with Fourier transform (as a generalized function) in $G$. But for Lebesgue-measurable sets $G$ the formula (2.1) is false in general (see [9]).

We prove (a); actually, its special variant for $\theta = 0$, since the arguments are essentially the same in the general case.

Clearly $P_\alpha(S_h^m(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\alpha > 0$, is a finite-dimensional space, and it is not hard to verify that

$$\dim P_\alpha(S_h^m(\mathbb{R}) \cap L_p(\mathbb{R})) \leq 2([\alpha/h] + 1) + m.$$  

It follows now from the definition that

$$\dim(S_h^m(\mathbb{R}) \cap L_p(\mathbb{R}) \cap L_p(\mathbb{R})) \leq h^{-1}.$$  

Let $2\alpha \geq m + 1$. We show that for all such $\alpha$ and $0 < \varepsilon \leq 1$

$$K_\varepsilon(\alpha, S_h^m(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R})) \geq 2[\alpha/h] - m.$$  

Indeed, if not, then for some $\alpha_0 \geq h(m + 1)/2$ there is an $n < 2[\alpha/h] - m$ such that $d_\varepsilon(P_\alpha(S_h^m(\mathbb{R}) \cap BL_p(\mathbb{R})), L_p(\mathbb{R})) < \varepsilon$. We consider the subspace

$$S(\alpha_0) = \{x(\cdot) \in S_h^m(\mathbb{R}) \mid \text{supp } x(\cdot) \subset [-h[\alpha_0/h], h[\alpha_0/h]]\}.$$  

Then $\dim S(\alpha_0) = 2[\alpha_0/h] - m$, and by the theory on the width of a ball (see, for example, [10], §8.1) we have

$$1 = d_\varepsilon(S(\alpha_0) \cap BL_p(\mathbb{R}), L_p(\mathbb{R})) \leq d_\varepsilon(P_\alpha(S_h^m(\mathbb{R}) \cap BL_p(\mathbb{R})), L_p(\mathbb{R})) < \varepsilon.$$  

This contradiction proves (2.2), which implies that

$$\dim(S_h^m(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R})) \geq h^{-1}.$$  

2.2. Let $1 \leq p \leq \infty$ and $r \in \mathbb{N}$. Denote by $W_p^r(\mathbb{R})$ the Sobolev space of functions $x(\cdot) \in L_p(\mathbb{R})$ for which the $(r - 1)$st derivative $x^{(r-1)}(\cdot)$ is locally absolutely continuous and $x^{(r)}(\cdot) \in L_p(\mathbb{R})$. This is a Banach space with the norm $\|x(\cdot)\|_{L_p(\mathbb{R})} + \|x^{(r)}(\cdot)\|_{L_p(\mathbb{R})}$.

We consider the table

$$T = \begin{pmatrix} \cdots & e_je_{j+1} & \cdots \\ \cdots & t_je_{j+1} & \cdots \end{pmatrix},$$

where $(t_j)_{j \in \mathbb{Z}} \in \Xi$, and $e_j$ ($j \in \mathbb{Z}$) is a nonempty subset of $\{0, 1, \ldots, r - 1\}$. With this table we associate the set $A := A(T)$ of functionals $y^* \in (W_p^r(\mathbb{R}))^*$ acting according to the rule $(y^*, y(\cdot)) = y^{(k)}(t_j)$, where $k_j \in e_j$, $j \in \mathbb{Z}$. The corresponding space $L(A)$ belongs to $\text{Lin}_1(W_p^r(\mathbb{R}))$. Indeed,

$$\text{card}\{y^* \in A | y^* \notin (W_p^r(\mathbb{R}))^*_\alpha\} = \sum_{\{j \in \mathbb{Z} | t_j \in (-\alpha, \alpha)\}} \text{card } e_j < \infty.$$  

From this we get, in particular,
Lemma 2.2. Suppose that 0 ≤ k ≤ r - 1, \( e_j = \{0, 1, \ldots, k\}, j \in \mathbb{Z}, \) and \( t_j = \theta + jh, j \in \mathbb{Z}, h > 0, \theta \in [0, h). \) In this case if \( L(A) \) is the corresponding subspace, then \( \operatorname{codim}(L(A), \mathbb{R}^r) = (k + 1)h^{-1}. \)

§3. Definitions of the \( \varphi \)-mean \( \nu \)-widths

Suppose that \( 1 \leq p \leq \infty, \) \( C \) is a centrally symmetric subset of \( L_p(\mathbb{R}), \varphi(\cdot) \in \Phi, \) and \( \nu > 0. \)

The \( \varphi \)-mean Kolmogorov \( \nu \)-width of the set \( C \) in \( L_p(\mathbb{R}) \) is defined to be the quantity
\[
d_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) := \inf_{L} \sup_{x(\cdot) \in C} \inf_{y(\cdot) \in L} \|x(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})},
\]
where the infimum is over all subspaces \( L \in \operatorname{Lin}_c(L_p(\mathbb{R})) \) such that
\[
\dim(L, L_p(\mathbb{R}), \varphi(\cdot)) < \nu.
\]

A subspace \( L \) on which this infimum is attained is said to be extremal for \( d_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)). \)

The \( \varphi \)-mean linear \( \nu \)-width of the set \( C \) in \( L_p(\mathbb{R}) \) is defined to be
\[
\lambda_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) := \inf_{(Y, \Lambda) \in C} \sup_{\varphi(\cdot) \in \varphi(\cdot)} \|x(\cdot) - \Lambda x(\cdot)\|_{L_p(\mathbb{R})},
\]
where the infimum is over all pairs \( (Y, \Lambda) \) such that \( Y \) is a normed space, \( Y \rightarrow L_p(\mathbb{R}), C \subset Y, \Lambda \in \mathcal{L}(Y, L_p(\mathbb{R})), \) \( \operatorname{Im}\Lambda \in \operatorname{Lin}_c(L_p(\mathbb{R})), \) and
\[
\dim(\operatorname{Im}\Lambda, L_p(\mathbb{R}), \varphi(\cdot)) < \nu.
\]

A pair on which this infimum is attained is said to be extremal for \( \lambda_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)). \)

It follows at once from the definitions that
\[
(3.1) \quad d_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) \leq \lambda_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)).
\]

The \( \varphi \)-mean Bernstein \( \nu \)-width of the set \( C \) in \( L_p(\mathbb{R}) \) is defined to be
\[
b_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) := \sup_{L} \sup_{\lambda > 0} \{\lambda > 0 | L \cap \lambda B L_p(\mathbb{R}) \subset C\},
\]
where the supremum is over all subspaces \( L \in \operatorname{Lin}_c(L_p(\mathbb{R})) \) such that
\[
\dim(L, L_p(\mathbb{R}), \varphi(\cdot)) > \nu \quad \text{and} \quad d_\nu(L \cap B L_p(\mathbb{R}), L_p(\mathbb{R}), \varphi(\cdot)) = 1.
\]

The last condition means that an analogue of the Tikhomirov theorem on the width of a ball must hold for \( L. \) It will be shown below (Theorem 5.4) that, in particular, spaces of splines and entire functions satisfy this requirement.

It follows easily from the definitions and obvious properties of the \( \varphi \)-mean Kolmogorov \( \nu \)-width that
\[
(3.2) \quad b_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) \leq d_\nu(C, L_p(\mathbb{R}), \varphi(\cdot)).
\]

The \( \varphi \)-mean Gel'fand \( \nu \)-width of the set \( C \) in \( L_p(\mathbb{R}) \) is defined to be
\[
d^\nu(C, L_p(\mathbb{R}), \varphi(\cdot)) := \inf_{(Y, L) \in C} \sup_{x(\cdot) \in L} \|x(\cdot)\|_{L_p(\mathbb{R})},
\]
where the infimum is over all parts \( (Y, L) \) such that \( Y \) is a normed space, \( Y \rightarrow L_p(\mathbb{R}), C \subset Y, L \in \operatorname{Lin}_c(Y), \) and \( \operatorname{codim}(L, Y, \varphi(\cdot)) \leq \nu \).
A pair on which this infimum is attained is said to be extremal for 
\( d^\nu(C, L_p(R), \varphi(\cdot)) \).

If \( \varphi(\alpha) = 2\alpha \), then we speak simply of the mean (Kolmogorov, linear, etc.) \( \nu \)-
width and denote the quantities introduced by \( d(\nu(C, L_p(R)), \tilde{T}_\nu(C, L_p(R)), \tilde{B}_\nu(C, L_p(R)), \) and \( \tilde{a}^\nu(C, L_p(R)) \), respectively.

Note that if we use the abstract definition of the \( \varphi \)-mean dimension or codimension (see \$1.1 and 1.3), then the analogues of the widths introduced above can be defined in a natural way in an arbitrary normed space.

\section*{4. Formulations of the Main Results}

Let \( 1 \leq p \leq \infty \) and \( r \in \mathbb{N} \). The space \( \mathcal{W}_p^r(R) \) was defined in \$2.2. Let

\[ W_p^r(R) := \{ x(\cdot) \in \mathcal{W}_p^r(R) : \| x^{(r)}(\cdot) \|_{L_p(R)} \leq 1 \}. \]

\textbf{Theorem 4.1.} Suppose that \( 1 \leq p = q \leq \infty \) or \( 1 \leq p < q < 2 \), \( r \in \mathbb{N} \), \( \varphi(\cdot) \in \Phi \), and \( \nu > 0 \). The quantity \( d(\nu(W_p^r(R), L_q(R), \varphi(\cdot)) \) is finite if and only if

\[ \liminf_{\alpha \to \infty} (\alpha / \varphi(\alpha)) < \infty. \]

Moreover, if \( \liminf_{n \to \infty} (\alpha / \varphi(\alpha)) > 0 \), then

\[ d(\nu(W_p^r(R), L_p(R), \varphi(\cdot)) = \begin{cases} \nu^{-r}, & 1 \leq p = q \leq \infty, \\ \nu^{-r+1/p-1/q}, & 1 \leq p < q \leq 2, \end{cases} \]

and the spaces \( S_m^{\nu}(R) \cap L_q(R), \ m \geq r - 1, \) and \( \mathcal{E}_\nu(R) \cap L_q(R) \) are asymptotically extremal for \( d(\nu(W_p^r(R), L_q(R), \varphi(\cdot)) \).

\textbf{Theorem 4.2.} Suppose that \( 1 \leq p \leq \infty \), \( r \in \mathbb{N} \), and \( \nu > 0 \). Then

\[ h(\nu(W_p^r(R), L_p(R))) = h(\nu(W_p^r(R), L_p(R))) = \tilde{a}(\nu(W_p^r(R), L_p(R))) = A(p, r)^{-\nu}, \]

where \( A(p, r) = 2^{-\nu} \| x(\cdot) \|_{L_p([0,1])} \) and \( x(\cdot) \) is the unique solution of the extremal problem\(^{(1)}\)

\[ \| x(\cdot) \|_{L_p([0,1])} \to \sup, \| x^{(i)}(\cdot) \|_{L_p([0,1])} \leq 1, \]

\[ x^{(i)}((1 - (-1)^i)/2) = 0, \quad 0 \leq i \leq r - 1. \]

In particular,

\[ A(1, r) = A(\infty, r) = \pi^{-r}K_r \quad \left( K_r := \frac{4}{\pi} \sum_{j=0}^\infty (-1)^j(2j + 1)^{-r+1} \right) \]

\[ A(2, r) = \pi^{-r}, \quad A(p, 1) = p \sin \frac{\pi}{p} / 2^{p(p - 1)^{1/p}}, \quad 1 < p < \infty. \]

The space \( S_1^{\nu, 1}(R) \cap L_p(R) \) is extremal for \( d(\nu(W_p^r(R), L_p(R)) \) when \( 1 \leq p \leq \infty \).

But if \( p = 1, 2, \) or \( \infty \), then the space \( S_1^{\nu, 1}(R) \cap L_p(R) \) is also extremal, and the spaces \( \mathcal{E}_\nu(R) \cap L_p(R) \) are also extremal for all \( m \geq r - 1. \)

The pair \( (\mathcal{W}_p^r(R), \Lambda) \), where the operator \( \Lambda \) associates with an \( x(\cdot) \in \mathcal{W}_p^r(R) \) the unique element in \( S_1^{\nu, 1}(R) \cap L_p(R) \) that interpolates \( x(\cdot) \) at the points \( t_k = k/\nu + (1 - (-1)^i)/4\nu, \ k \in \mathbb{Z}, \) is extremal for \( \tilde{a}(\nu(W_p^r(R), L_p(R))) \). \( 1 \leq p \leq \infty. \)

\(^{(1)}\)Here \( \var_{p, 1} \| x^{(r-1)}(\cdot) \|_{L_p([0,1])} \) should be understood in place of \( \| x^{(r)}(\cdot) \|_{L_p([0,1])} \).
The pair \((\mathcal{H}_p'(R), H)\), where \(H\) is given by \(FHx(\cdot) := \chi_{x\nu}(\cdot)Fx(\cdot)\) \((F\) is the Fourier transform acting in \(L_2(R)\), and \(\chi_{x\nu}(\cdot)\) is the characteristic function of the interval \([-\pi\nu, \pi\nu]\), is extremal for \(\overline{\Lambda}_\nu(W_2^*(R), L_2(R))\).

The pair \((\mathcal{H}_p'(R), L)\), where \(L = \{x(\cdot) \in \mathcal{H}_p'(R) | x(j/\nu) = 0, j \in \mathbb{Z}\}\), is extremal for \(d^*(W_p'(R), L_p(R)), 1 \leq p \leq \infty\).

§ 5. Preliminary results

We begin with an assertion relating to a lower estimate for the \(q\)-mean Kolmogorov \(\nu\)-width.

Recall (see 1.1) that if \(x(\cdot) \in L_p(R), 1 \leq p \leq \infty\), and \(\alpha > 0\), then \(P_\alpha x(t) = x(t)\) for a.e. \(t \in [-\alpha, \alpha]\) and \(P_\alpha x(t) = 0\) for a.e. \(t \not\in [-\alpha, \alpha]\).

**Theorem 5.1.** Suppose that \(1 \leq p \leq \infty, L \in \text{Lin}(L_p(R)), \phi(\cdot) \in \Phi, \nu > 0,\) and for each \(\alpha > 0\) there exist \(S(\alpha) \in \text{Lin}(P_\alpha(L))\) and \(\Lambda_\alpha \in \mathcal{L}(S(\alpha), L)\) such that \(\dim S(\alpha) < \infty, P_\alpha \circ \Lambda_\alpha = \text{the identity operator},\) and \(\liminf_{\alpha \to \infty} (\dim S(\alpha)/\phi(\alpha)) > \nu.\) Then \(d_\nu(L \cap BL_p(R), L_p(R), \phi(\cdot)) \leq \liminf_{\alpha \to \infty} \|\Lambda_\alpha\|^{-1}\).

**Proof.** Suppose that \(M \in \text{Lin}(L_p(R)), \dim(M, L_p(R), \phi(\cdot)) \leq \nu,\) and \(\epsilon > 0,\) For each \(\alpha > 0\) there exists by definition a \(Q := Q(\epsilon, \alpha) \in \text{Lin}(L_p(R))\) such that \(\dim Q = K_\epsilon(\alpha, M, L_p(R))\) and \(d(P_\alpha y(\cdot), Q, L_p(R)) < \epsilon \|y(\cdot)\|_{L_p(R)}\) for all \(y(\cdot) \in M\). Suppose that \(\alpha > 0\) and \(x(\cdot) \in S(\alpha) \cap BL_p(R).\) Considering the last inequality, the subadditivity of the function of distance to a subspace, the properties of the operator \(P_\alpha,\) and the conditions of the theorem, we have for any \(y(\cdot) \in M\) that

\[
\|\Lambda_\alpha x(\cdot) - y(\cdot)\|_{L_p(R)} \geq \|P_\alpha \circ \Lambda_\alpha x(\cdot) - P_\alpha y(\cdot)\|_{L_p(R)}
\]

\[
= \|x(\cdot) - P_\alpha y(\cdot)\|_{L_p(R)} \geq d(x(\cdot) - P_\alpha y(\cdot), Q, L_p(R))
\]

\[
\geq d(x(\cdot), Q, L_p(R)) - d(P_\alpha y(\cdot), Q, L_p(R))
\]

\[
\geq d(x(\cdot), Q, L_p(R)) - \epsilon \|y(\cdot)\|_{L_p(R)}
\]

\[
\geq d(x(\cdot), Q, L_p(R)) - \epsilon \|\Lambda_\alpha x(\cdot) - y(\cdot)\|_{L_p(R)} - \epsilon \|\Lambda_\alpha x(\cdot)\|_{L_p(R)}.
\]

i.e.,

\[(1 + \epsilon)\|\Lambda_\alpha x(\cdot) - y(\cdot)\|_{L_p(R)} \geq d(x(\cdot), Q, L_p(R)) - \epsilon \|\Lambda_\alpha x(\cdot)\|_{L_p(R)}.
\]

Consequently,

\[(5.1) \quad (1 + \epsilon) d(\Lambda_\alpha x(\cdot) M, L_p(R)) \geq d(x(\cdot), Q, L_p(R)) - \epsilon \|\Lambda_\alpha x(\cdot)\|_{L_p(R)}.
\]

Since \(x(\cdot) \in S(\alpha) \cap BL_p(R),\) it follows that \(\Lambda_\alpha x(\cdot) \in L \cap \|\Lambda_\alpha\| BL_p(R),\) and hence, by (5.1),

\[(5.2) \quad (1 + \epsilon)\|\Lambda_\alpha\| d(L \cap BL_p(R), M, L_p(R)) \geq d(S(\alpha) \cap BL_p(R), Q, L_p(R)) - \epsilon \|\Lambda_\alpha\|.
\]

Let \(\{\alpha_n\}_{n \in \mathbb{N}}\) be a sequence such that

\[
\liminf_{n \to \infty} (K_\epsilon(\alpha_n, M, L_p(R))/\phi(\alpha_n)) = \lim_{n \to \infty} (K_\epsilon(\alpha_n, M, L_p(R))/\phi(\alpha_n)).
\]

By assumption, there exists a \(\nu_1 \leq \liminf_{n \to \infty} (\dim S(\alpha)/\phi(\alpha))\) with \(\nu_1 > \nu,\) and hence \(\nu_1 - \rho > \nu + \rho\) for some \(\rho > 0.\) Then there exists an \(n_0 \in \mathbb{N}\) such that \(K_\epsilon(\alpha_n, M, L_p(R)) \leq (\nu + \rho)\phi(\alpha_n)\) and \(\dim S(\alpha_n)/\phi(\alpha_n) \geq (\nu_1 - \rho)\phi(\alpha_n)\) for all \(n \geq n_0,\) and thus

\[
\dim S(\alpha_n) \geq (\nu_1 - \rho)\phi(\alpha_n) > (\nu + \rho)\phi(\alpha_n) \geq K_\epsilon(\alpha_n, M, L_p(R)) = \dim S(\alpha_n, \epsilon).
\]
By the theorem on the width of a ball,
\[ d(S(\alpha_n) \cap B_{L_p}(R), Q(\alpha_n, \varepsilon), L_p(R)) \geq 1 \]
for all \( n \geq n_0 \), and then it follows from (5.2) that
\[ (1 + \varepsilon)d(L \cap B_{L_p}(R), M, L_p(R)) \geq \|A_n\|^{-1} - \varepsilon. \]

From this the required assertion now follows easily. The theorem is proved.

It is clear that the conclusion of the theorem remains in force if all its conditions are required only for sufficiently large \( n \).

We remark that this result carries over in an obvious way to the \( \phi \)-mean Kolmogorov \( \nu \)-width in an arbitrary normed space.

The following assertion concerns the precise value of the Bernstein widths for classes of periodic functions.

Let \( T \) be the unit circle, which will be realized as the interval \([-\pi, \pi]\) with endpoints identified. For each \( 1 \leq p \leq \infty \) and \( r \in \mathbb{N} \) denote by \( \mathcal{H}_r^p(T) \) the collection of functions \( x(\cdot) \) on \( T \) for which the \((r - 1)\)st derivative \( x^{(r-1)}(\cdot) \) is absolutely continuous and \( x^{(r)}(\cdot) \in L_p(T) \).

A function \( x(\cdot) \in \mathcal{H}_r^p(T) \) has the representation
\[ x(t) = a + \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(t - \tau) u(\tau) \, d\tau, \]
where \( a \in \mathbb{R}, B_r(t) = \sum_{k=1}^{\infty} k^{-r} \cos(kt - \pi r/2), u(\cdot) \in L_p(T), \int_{-\pi}^{\pi} u(\tau) \, d\tau = 0, \) and, further \( x^{(r)}(t) = u(t) \) a.e.

Let \( W_r^p(T) := \{ x(\cdot) \in \mathcal{H}_r^p(T) | \| x^{(r)}(\cdot) \|_{L_p(T)} \leq 1 \} \).

**Theorem 5.2.** Suppose that \( 1 < p < \infty \) and \( r, n \in \mathbb{N} \). Then
\[ b_{2n-1}(W_r^p(T), L_p(T)) = A(p, r)/n^r, \]
where \( A(p, r) \) is defined in Theorem 4.2. An extremal space is given by the \( 2n \)-dimensional space \( T_{2n}(p, r) \subset W_r^p(T) \) formed by the functions \( x(\cdot) \) of the form
\[ x(t) = a + \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(t - \tau) \left( \sum_{j=1}^{2n} b_j \chi_j(\tau) x_n^{(r)}(\tau - (j - 1)\pi/n) \right) \, d\tau, \]
where \( a, b_1, \ldots, b_{2n} \in \mathbb{R}, \sum_{j=1}^{2n} b_j = 0, \chi_j(\cdot) \) is the characteristic function of the interval \( \Delta_j = [-\pi - (j - 1)\pi/n, \pi + j\pi/n], \) \( 1 \leq j \leq 2n, \) and the function \( x_n(\cdot) \) is such that \( x_n(t) = -x_n(t - \pi/n) \) for all \( t \in \mathbb{T}, \)
\[ x_n(t) := \begin{cases} (2\pi)^{-1/p} (2n/\pi)^{-r} \tilde{x}(2nt/\pi), & 0 \leq t \leq \pi/2n, \\ (2\pi)^{-1/p} (2n/\pi)^{-r} \tilde{x}(-2nt/\pi + 2), & \pi/2n \leq t \leq \pi/n, \end{cases} \]
with \( \tilde{x}(\cdot) \) the solution of the extremal problem (4.1).

**Proof.** For \( p = \infty \) this fact was established by Tikhomirov [12], and \( T_{2n}(\infty, r) \) is the space of periodic \( r \)-th order splines of defect 1 with nodes at the points \( k\pi/n, k = -n + 1, \ldots, n. \)

Let \( 1 < p < \infty \). To prove the theorem it suffices to establish the inequality
\[ \| x^{(r)}(\cdot) \|_{L_p(T)} \leq \frac{n^r}{A(p, r)n^r} \| x(\cdot) \|_{L_p(T)} \]
for all \( x(\cdot) \in T_{2n}(p, r) \). Indeed, the required lower estimate for \( b_{2n-1}(W_r^p(T), L_p(T)) \) follows at once from (5.3). In [13] Buslaev and Tikhomirov showed that
\[ d_{2n-1}(W_r^p(T), L_p(T)) = A(p, r)n^r / n^r, \]
and the necessary upper estimate is a consequence of the fact that
\[ b_{2n-1}(W_p'(T),L_p(T)) \leq d_{2n-1}(W_p'(T),L_p(T)). \]

The proof of (5.3) is based on the following result.

**Theorem** (Buslaev and Tikhomirov [13]). Suppose that \( 1 < p < \infty , r, n \in \mathbb{N} \), \( \lambda_n = (2n/\pi)^r \| x(\cdot) \|_{L_p([0,1])}^{-1} \), where \( x(\cdot) \) is a solution of the problem (4.1), and the function \( x_n(\cdot) \) is defined in Theorem 5.2. Then the pair \((\lambda_n, x(\cdot))\) and only it (up to translates of \( x_n(\cdot)\)) satisfies the differential equation
\[
(5.4) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(t-\tau)|x_n(\tau)|^{p-1} \text{sgn} x_n(\tau) \, d\tau = (-1)^r \lambda_n^{-p} |x_n^{(r)}(\cdot)|^{p-1} \text{sgn} x_n^{(r)}(\cdot)
\]
and the relations
\[
(5.5) \quad \int_{-\pi}^{\pi} x_n^{(r)}(t) \, dt = 0, \quad \| x_n^{(r)}(\cdot) \|_{L_p(T)} = 1.
\]

Consider the extremal problem
\[
(5.6) \quad \| x(\cdot) \|_{L_p(T)}^p / \| x^{(r)}(\cdot) \|_{L_p(T)}^p \to \inf, \quad x(\cdot) \in T_{2n}(p, r),
\]
and denote by \( \alpha^p \) its value.

We multiply both sides of (5.4) by \( x_n^{(r)}(t) \) and integrate over \( T \), getting that \( \lambda_n = \| x_n(\cdot) \|_{L_p(T)}^{-1} \) or \( \lambda_n^p = \| x_n^{(r)}(\cdot) \|_{L_p(T)}^p / \| x_n(\cdot) \|_{L_p(T)}^p \), i.e., \( \lambda_n^{-1} \geq \alpha^p \).

The theorem will be proved if we show that \( \lambda_n^{-1} = \alpha^p \). Note first that for \( x(\cdot) \in T_{2n}(p, r) \) we have
\[
\| x^{(r)}(\cdot) \|_{L_p(T)} = \sum_{i=1}^{2n} \int_{\Delta_i} \left| \sum_{j=1}^{2n} b_j x_n^{(r)} \left( t - \frac{(j - 1)\pi}{n} \right) \right|^p \, dt
\]
\[
= \sum_{i=1}^{2n} |b_i|^p \int_{\Delta_i} |x_n^{(r)}(t)|^p \, dt = \frac{1}{2n} \sum_{i=1}^{2n} |b_i|^p.
\]
Then the problem (5.6) can be written in the form
\[
(5.7) \quad \left( a + \sum_{j=1}^{2n} b_j z_j(\cdot) \right) \left. \frac{1}{L_p(T)} \right| \left. \frac{1}{L_p(T)} \right| \frac{1}{2n} \sum_{j=1}^{2n} |b_j|^p \to \inf, \quad \sum_{j=1}^{2n} b_j = 0, \quad a \in \mathbb{R},
\]
where
\[
z_j(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(t-\tau) x_j(\tau) x_n^{(r)}(\tau - (j - 1)\pi/n) \, d\tau
\]
and \( x_j(\cdot) \) is the characteristic function of the interval \( \Delta_j \), \( 1 \leq j \leq 2n \).

This is a smooth finite-dimensional problem that clearly has a solution \((a, b_1, \ldots, b_{2n})\) and \((b_1, \ldots, b_{2n}) \neq 0\). According to the Lagrange multiplier rule, there exists a \( \mu \in \mathbb{R} \) such that the derivatives of the function \( (a, b_1, \ldots, b_{2n}) \to g(a, b_1, \ldots, b_{2n}) + \mu(b_1 + \cdots + b_{2n}) \) (where \( g(\cdot) \) is the function being minimized in (5.7)) with respect to \( a, b_1, \ldots, b_{2n} \) at the point \((\overline{a}, \overline{b}_1, \ldots, \overline{b}_{2n})\) are equal to zero. This leads to the relations
\[
(5.8) \quad \int_{-\pi}^{\pi} |\overline{x}(t)|^{p-1} \text{sgn} \overline{x}(t) \, dt = 0,
\]
\[
\int_{-\pi}^{\pi} z_i(t)|x(t)|^{p-1} \operatorname{sgn} \overline{x}(t) \, dt
\]

\(\frac{\alpha \beta}{2n} |\overline{b}_i|^{p-1} \operatorname{sgn} \overline{b}_i + \frac{\mu}{2n} \sum_{j=1}^{2n} |\overline{b}_j|^p, \quad i = 1, \ldots, 2n,\)

where \(\overline{x}(\cdot) = \sum_{i=1}^{2n} \overline{b}_i z_i(\cdot)\).

We remark that \(g(a, b_1, \ldots, b_{2n}) = g(c \alpha, c b_1, \ldots, c b_{2n})\) for any \(c \neq 0\), and hence the vector \((c \alpha, c b_1, \ldots, c b_{2n})\) is also a solution of \((5.7)\). Thus, it can be assumed that \(|\overline{b}_i| \leq 1, \quad i = 1, \ldots, 2n,\) and \(\overline{b}_{i_0} = (-1)^{i_0+1}\) for some \(1 \leq i_0 \leq 2n\).

It follows from the definitions of \(x_n(\cdot)\) and \(\overline{x}(\cdot)\) that

\[x_n^{(r)}(t) - \overline{x}^{(r)}(t) = \sum_{i=1}^{2n} ((-1)^{i+1} - \overline{b}_i) x_i(t) x_n^{(r)} \left( t - \frac{(i-1)\pi}{n} \right),\]

and hence \(P(x_n^{(r)}(\cdot), \overline{x}^{(r)}(\cdot))\) has at most \(2n - 2\) sign changes. Then, by Rolle's theorem, \(P(x_n(\cdot) - \overline{x}(\cdot)) \leq 2n - 2\), and thus

\[P(|x_n(\cdot)|^{p-1} \operatorname{sgn} x_n(\cdot) - |\overline{x}(\cdot)|^{p-1} \operatorname{sgn} \overline{x}(\cdot)) \leq 2n - 2.\]

If we now multiply both sides of \((5.4)\) by \(x_n^{(r)}(t - (i-1)\pi/n), \quad 1 \leq i \leq 2n,\) and integrate over the interval \(\Delta_i\), we get

\[\int_{\Delta_i} x_n^{(r)} \left( t - \frac{(i-1)\pi}{n} \right) \left( \frac{1}{\pi} \int_{-\pi}^{\pi} B_s(t - \tau)(|x_n(\tau)|^{p-1} \operatorname{sgn} x_n(\tau) - |\overline{x}(\tau)|^{p-1} \operatorname{sgn} \overline{x}(\tau)) \, d\tau \right) \, dt
\]

\[= (-1)^r \int_{-\pi}^{\pi} z_i(t) |x_n(t)|^{p-1} \operatorname{sgn} x_n(i) - |\overline{x}(t)|^{p-1} \operatorname{sgn} \overline{x}(t) \, dt
\]

Denote by \(f(\cdot)\) the factor multiplying \(x_n^{(r)}(t - (i-1)\pi/n)\) in the integral on the left-hand side of this equality. If we then assume that \(\lambda_n^{-1} > \alpha\), then we arrive at the relations

\[\operatorname{sgn} \int_{\Delta_i} x_n^{(r)} \left( t - \frac{(i-1)\pi}{n} \right) f(t) \, dt = (-1)^{r+i+1}, \quad i = 1, \ldots, 2n.\]

Suppose for definiteness that \(x_n^{(r)}(t - (i-1)\pi/n) > 0\) interior to \(\Delta_i, \quad i = 1, \ldots, 2n.\) Then it follows from \((5.12)\) that there are points \(t_i \in \Delta_i\) such that \(\operatorname{sgn} f(t_i) = (-1)^{r+i+1}, \quad i = 1, \ldots, 2n,\) i.e., \(P(f(\cdot)) \geq 2n - 1.\) But \(f(\cdot)\) is periodic, and hence \(P(f(\cdot)) \geq 2n\); therefore, \(P(f(\cdot)) = 2n.\) Further,

\[f^{(r)}(\cdot) = \{x_n(\cdot)|^{p-1} \operatorname{sgn} x_n(\cdot) - |\overline{x}(\cdot)|^{p-1} \operatorname{sgn} \overline{x}(\cdot),\]

because in view of \((5.8)\) and the definition of \(x_n(\cdot)\) the function \(|x_n(\cdot)|^{p-1} \operatorname{sgn} x_n(\cdot) - |\overline{x}(\cdot)|^{p-1} \operatorname{sgn} \overline{x}(\cdot)\) is equal to zero in the mean. We have arrived at a contradiction.
to (5.10), and hence $\lambda_n^{-1} \leq \alpha$. This means that $\lambda_n^{-1} = \alpha$, and the inequality (5.3) is proved for $1 < p < \infty$.

**Theorem 5.3.** Suppose that $1 < p \leq \infty$, $r \in \mathbb{N}$, and $h > 0$. There exists a subspace $T_h(p, r) \subseteq \mathscr{H}'(\mathbb{R})$ such that

$$T_h(p, r) \subseteq \text{Lin}_c(L_p(\mathbb{R})), \quad \overline{\text{dim}}(T_h(p, r), L_p(\mathbb{R})) = h^{-1},$$

and for all $\chi(\cdot) \in T_h(p, r)$

$$\|\chi^{(r)}(\cdot)\|_{L_p(\mathbb{R})} \leq \frac{h^{-r}}{A(p, r)} \|\chi(\cdot)\|_{L_p(\mathbb{R})},$$

where $A(p, r)$ is defined in Theorem 4.2.

**Proof.** If $p = \infty$, then this is a rephrasing of a well-known result (see [14]), and in this case $T_h(p, r) = \mathscr{S}'(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

Suppose that $1 < p < \infty$ and $n \in \mathbb{N}$ is such that $2n \geq r + 1$. Denote by $M(n)$ the collection of $\chi(\cdot)$ such that $\chi(nh/\pi) \in T_{2n}(p, r)$ (see Theorem 5.2), and, further, $\chi^{(i)}(nh) = 0$, $i = 0, 1, \ldots, r - 1$. The last equalities and the condition $\sum_{i=0}^{r-1} b_i = 0$ give a system of $r + 1$ homogeneous linear equations in the $2n + 1$ unknowns $a, b_1, \ldots, b_{2n}$ (see the definition of $T_{2n}(p, r)$), and thus $\dim M(n) = 2n - m$, where $0 \leq m \leq r$.

Let $\chi(\cdot) \in M(n)$. Then $\chi(nh/\pi)$ satisfies (5.3). Extending this function by zero outside of $[-\pi, \pi]$, we can assume that it belongs to $\mathscr{H}'(\mathbb{R})$. After a change of variables we now get that (5.13) holds for $M(n)$.

Let $T_h(p, r) = U\{M(n)|2n \geq r + 1\}$. Obviously, $T_h(p, r)$ is a subspace of $\mathscr{H}'(\mathbb{R})$ satisfying (5.13).

The proof that $\overline{\text{dim}}(T_h(p, r), L_p(\mathbb{R})) = h^{-1}$ is carried out according to the same scheme as the proof of Lemma 2.1(a).

We now show that the analogue of the theorem on the width of a ball is valid for the concrete spaces considered above.

**Theorem 5.4.** Assume that $1 \leq p \leq \infty$ and $\nu > 0$. Then:

(a) If $m \in \mathbb{Z}_+$, $\zeta \in \Xi$, $\varphi(\cdot) \in \Phi$, and $\dim(S_\zeta^m(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R}), \varphi(\cdot)) > \nu$, then

$$d_{\nu}(S_\zeta^m(\mathbb{R}) \cap BL_p(\mathbb{R}), L_p(\mathbb{R}), \varphi(\cdot)) = 1.$$

(b) If $\sigma > 0$ and $\overline{\text{dim}}(S_{\sigma}^m(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R})) > \nu$, then

$$\overline{d}_{\nu}(S_{\sigma}^m(\mathbb{R}) \cap BL_p(\mathbb{R}), L_p(\mathbb{R})) = 1.$$

(c) If $h > 0$ and $\overline{\text{dim}}(T_h(p, r), L_p(\mathbb{R})) > \nu$, then

$$\overline{d}_{\nu}(T_h(p, r) \cap BL_p(\mathbb{R}), L_p(\mathbb{R})) = 1.$$

**Proof.** (a) We assume for simplicity that $\varphi(\alpha) = 2\alpha$ and $S_\zeta^m(\mathbb{R}) = S_\zeta^m(\mathbb{R}), h > 0$. Then $h^{-1} > \nu$ by Lemma 2.1 and the condition of the theorem. Let $S(\alpha)$ be the same as in Lemma 2.1; then

$$\lim_{\alpha \to \infty} \left(\frac{\dim S(\alpha)}{2\alpha}\right) = h^{-1} > \nu.$$ 

Further, let $\Lambda_\alpha$ be the operator extending functions by zero outside of $[-\alpha, \alpha]$. It is clear that $\|\Lambda_\alpha\| = 1$, $\alpha > 0$. Consequently, by Theorem 5.1,

$$\overline{d}_{\nu}(S_\zeta^m(\mathbb{R}) \cap BL_p(\mathbb{R}), L_p(\mathbb{R})) \geq 1.$$

The converse inequality is obvious.
(b) According to Lemma 2.1, the condition of the theorem means that $\sigma/\pi > \nu$. Suppose that $0 < \beta < \beta_1 < \sigma$ and $(\sigma - \beta_1)/\pi > \nu$. Let $t_j = j\pi(\sigma - \beta)$, and let

$$
\varphi_j(t) = \frac{\sin(\sigma - \beta)(t - t_j)\sin \beta(t - t_j)}{\beta(\sigma - \beta)(t - t_j)}, \quad j \in \mathbb{Z}.
$$

It is clear that $\varphi_j(\cdot) \in \mathcal{E}_\alpha^p(\mathbb{R}) \cap L_p(\mathbb{R})$.

We associate with each $\alpha > 0$ the space

$$
S(\alpha) = \left\{ \varphi_j(\cdot) \right\}_{|j| \leq [\alpha(\sigma - \beta_1)/\pi]}.
$$

Then there exists an $\alpha_0 > 0$ such that for each $\alpha \geq \alpha_0$ and all $x(\cdot) \in S(\alpha)$

$$
\|x(\cdot)\|_{L_p(\mathbb{R})} \leq \gamma(\alpha)\|x(\cdot)\|_{L_p(\mathbb{R})},
$$

where $\gamma(\alpha) > 0$ and $\gamma(\alpha) \to 1$ as $\alpha \to \infty$.

The proof reduces essentially to a direct estimate of the norm of a function in $S(\alpha)$, and hence we omit it.

If $\Lambda_\alpha$ denotes the operator associating with a function of the form $x(\cdot)\chi_{\alpha}(\cdot)$, where $x(\cdot) \in S(\alpha)$ and $\chi_{\alpha}(\cdot)$ is the characteristic function of $[-\alpha, \alpha]$, its unique extension (because of the analyticity of $x(\cdot)$), then it follows from (5.12) that $\|\Lambda_\alpha\| \leq \gamma(\alpha)$.

It is then clear that $\dim S(\alpha) = 2[\alpha(\sigma - \beta_1)/\pi] + 1$, and since

$$
\lim_{\alpha \to \infty} (\dim S(\alpha)/2\alpha) = (\sigma - \beta_1)/\pi > \nu,
$$

it follows by Theorem 5.1 that

$$
\overline{d}_p(\mathcal{E}_\alpha^p(\mathbb{R}) \cap B; L_p(\mathbb{R}), L_p(\mathbb{R})) \geq 1.
$$

(c) The proof of this assertion actually repeats that of (a), and thus we omit it.

**Theorem 5.5.** Suppose that $1 \leq p \leq \infty$, $r \in \mathbb{N}$, and $h > 0$. For any $x(\cdot) \in W_p^r(\mathbb{R})$ there exists a unique spline $s(x(\cdot), \cdot) \in S_n^{-1}(\mathbb{R}) \cap L_p(\mathbb{R})$ such that

$$
s(x(\cdot), kh + (1 - (-1)^r)h/4) = x(kh + (1 - (-1)^r)h/4), \quad k \in \mathbb{Z},
$$

and thus

$$
\|x(\cdot) - s(x(\cdot), \cdot)\|_{L_p(\mathbb{R})} \leq A(p, r)h^r,
$$

where $A(p, r)$ is defined in Theorem 4.2.

**Proof.** Let $x(\cdot) \in W_p^r(\mathbb{R})$. Denote by $\eta(\cdot)$ an infinitely differentiable function on $\mathbb{R}$ such that $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$, supp $\eta(\cdot) \subset [-1, 1]$, and $\eta(t) = 1$ for $t \in [-1/2, 1/2]$. Further, let $\eta_\theta(t) = \eta(t/\theta)$, where $\theta > 0$ and $x_\theta(\cdot) = x(\cdot)\eta_\theta(\cdot)$.

By the Leibniz formula,

$$
x_\theta^{(r)}(t) = \sum_{j=0}^{r-1} \binom{r}{j} x^{(j)}(t)\eta_\theta^{(r-j)}(t)\theta^{-(r-j)}.
$$

The Kolmogorov inequality for $p = \infty$ [15] and the Stein inequality for $1 \leq p < \infty$ [16] give us that the derivatives $x^{(j)}(\cdot)$, $j = 1, \ldots, r - 1$, are bounded in $L_p(\mathbb{R})$, and then it follows from (5.16) that

$$
\|x_\theta^{(r)}(\cdot)\|_{L_p(\mathbb{R})} \leq \rho(\theta) + \|x^{(r)}(\cdot)\|_{L_p(\mathbb{R})} \leq \rho(\theta) + 1,
$$

where $\rho(\theta) \to 0$ as $\theta \to \infty$.

Let $n \in \mathbb{N}$, and define $y_n(\cdot) := (p(nh) + 1)^{-1}x_{nh}(\cdot)$. It is clear that

$$
\text{supp} \ y_n(\cdot) \subset [-nh, nh] \quad \text{and} \quad \|y_n^{(r)}(\cdot)\|_{L_p([-nh, nh])} \leq 1.
$$
Then the function \( z_n(t) = (\pi/nh)^{1/2} y_n(nht/\pi) \) belongs to \( W_p(T) \), and by a theorem of Buslaev and Tikhomirov [13], there exists a unique spline \( \xi_n(t) \in S^{(r-1)}_n(T) \) (the space of periodic splines of defect 1 with order \( r - 1 \) and nodes at the points \( k\pi/n \), \( k = -n, \ldots, n - 1 \)), such that

\[
\xi_n(k\pi/n + (1 - (-1)^{r/4})\pi/4n) = z_n(k\pi/n + (1 - (-1)^{r/4})\pi/4n)
\]

for \( k = -n, \ldots, n - 1 \), and

\[
\|z_n(t) - \xi_n(t)\|_{L_p(T)} \leq A(p, r)n^{-1/2}.
\]

From this, a simple computation shows that the \( 2nh \)-periodic spline \( \xi_n(t) = (nh/\pi)^{-1/2} \xi_n(\pi t/nh) \) has the property that

\[
\xi_n((kh + (1 - (-1)^{r/4})h/4) = y_n(kh + (1 - (-1)^{r/4})h/4)
\]

for \( k = -n, \ldots, n - 1 \), and

\[
\|y_n(t) - \xi_n(t)\|_{L_p([-nh, nh])} \leq A(p, r)h^{1/2}.
\]

Since \( y_n(t) \) coincides with \( \rho(nh + 1)^{-1}x(t) \) on \([-nh/2, nh/2]\), for \( j \) such that \( n_j \geq n \) we have (by (5.17)) that

\[
\|x(t) - \xi_n(t)\|_{L_p([-nh/2, nh/2])} = \|(\rho(n_j/h) + 1)y_n(t) - \xi_n(t)\|_{L_p([-nh/2, nh/2])} \\
\leq \rho(n_j/h)y_n(t)\|_p \|y_n(t) - \xi_n(t)\|_{L_p([-nh/2, nh/2])} \\
\leq \rho(n_j/h)\|x(t)\|_{L_p(R)} + A(p, r)h^{1/2}.
\]

Passing here to the limit as \( j \to \infty \), and then as \( n \to \infty \) (and using Fatou’s lemma for \( 1 \leq p < \infty \), we get the estimate (5.15) for some spline \( s(t) \in S^{(r-1)}_h(R) \cap L_p(R) \). Obviously, \( s((kh + (1 - (-1)^{r/4})h/4), k \in \mathbb{Z} \). But since the sequence \( \{x((kh + (1 - (-1)^{r/4})h/4), k \in \mathbb{Z} \) is bounded (because of the boundedness of functions in \( W_p(R) \), the interpolation spline is unique (see, for example, [21]). The theorem is proved.

For \( p = 1, 2 \), and \( \infty \) this result was established earlier in [22], [23], and [24], respectively.

Suppose that \( 1 \leq p \leq \infty \), \( m \in \mathbb{N} \) and \( h > 0 \). Denote by \( \mathcal{O}_p^m(R, h) \) the collection of functions \( x(t) \in C^{m-1}(R) \) such that \( x^{(r-1)}(t) \) is locally absolutely continuous, \( x^{(r)}(t) \in L_p(R) \), and \( x(jh) = 0 \), \( j \in \mathbb{Z} \).

**Theorem 5.6.** Suppose that \( 1 \leq p \leq q \leq \infty \), \( k \in \mathbb{Z} \), \( m \in \mathbb{N} \), \( k < m \), and \( h > 0 \). Then there exists a constant \( c = c(k, m) > 0 \) such that for all \( x(t) \in \mathcal{O}_p^m(R, h) \)

\[
\|x^{(r)}(t)\|_{L_p(R)} \leq c h^{m-k-1/p+1/q}\|x^{(m)}(t)\|_{L_p(R)}.
\]

**Proof.** Suppose that \( x(t) \in \mathcal{O}_p^m(R, h) \), and \( j \in \mathbb{Z} \). Obviously, the function \( x^{(m-1)}(t) \) has a zero \( \theta \) on \([jh, (j+m)h]\). Then for all \( t \in [jh, (j+m)h] \) we have, by the Hölder inequality (\( 1 \leq p \leq \infty \) for definiteness), that

\[
\|x^{(m-1)}(t)\|_{L_p(R)} \leq \int_0^1 |x^{(m)}(\tau)| d\tau \leq (mh)^{1/p'} \left( \int_{jh}^{(j+m)h} |x^{(m)}(\tau)|^p d\tau \right)^{1/p}.
\]

For \( p = \infty \) this result was proved earlier by Tikhomirov [18], for \( p = 1 \) by Korneichuk [19], and for \( p = 2 \) by Melkman and Micchelli [20].
Raising the left-hand and right-hand sides of this inequality to the $q$th power (if $1 \leq q < \infty$), and then integrating over $[jh, (j + m)h]$, we get
\[
\int_{jh}^{(j+m)h} |x^{(m-1)}(t)|^q \, dt \leq (mh)^{q/p'+1} \left( \int_{jh}^{(j+m)h} |x^{(m)}(\tau)|^p \, d\tau \right)^{q/p}.
\]
Summing these inequalities over all $j \in \mathbb{Z}$ and using the inequality for the averages, we obtain
\[
m \int_{\mathbb{R}} |x^{(m-1)}(t)|^q \, dt \leq (mh)^{q/p'+1} \sum_{j \in \mathbb{Z}} \left( \int_{jh}^{(j+m)h} |x^{(m)}(\tau)|^p \, d\tau \right)^{q/p} \leq (mh)^{q/p'-1} \left( \sum_{j \in \mathbb{Z}} \int_{jh}^{(j+m)h} |x^{(m)}(\tau)|^p \, d\tau \right)^{q/p} = (mh)^{q/p'+1} m^{q/p} \left( \int_{\mathbb{R}} |x^{(m)}(\tau)|^p \, d\tau \right)^{q/p}.
\]
This proves (5.18) for $k = m - 1$, and, in particular, shows that $x(\cdot) \in \mathcal{W}_{q}^{m-1}(\mathbb{R}, h)$.
Repeating the arguments the necessary number of times (but now for coinciding metrics), we get (5.18). The theorem is proved.

The next result is a refinement of this inequality for compatible metrics.

**Theorem 5.7.** Suppose that $1 \leq p \leq \infty$, $m \in \mathbb{N}$, and $h > 0$. Then the sharp inequality
\[
\|x(\cdot)\|_{L_p(\mathbb{R})} \leq A(p, m) h^m \|x^{(m)}(\cdot)\|_{L_p(\mathbb{R})}
\]
holds for all $x(\cdot) \in \mathcal{W}_p^m(\mathbb{R}, h)$, where $A(p, m)$ was defined in Theorem 4.2.

Moreover, if $p = 1, 2, \text{ or } \infty$, $k \in \mathbb{N}$, and $k < m$, then the sharp inequality
\[
\|x^{(k)}(\cdot)\|_{L_p(\mathbb{R})} \leq A(p, m - k) h^{m-k} \|x^{(m)}(\cdot)\|_{L_p(\mathbb{R})}
\]
holds for all $x(\cdot) \in \mathcal{W}_p^m(\mathbb{R}, h)$.

**Proof.** Let $x(\cdot) \in \mathcal{W}_p^m(\mathbb{R}, h)$. By the preceding theorem, $x(\cdot) \in L_p(\mathbb{R})$, and hence $x(\cdot) \in \mathcal{W}_p^m(\mathbb{R})$ (it can be assumed that $x(\cdot) \in W_p^m(\mathbb{R})$). In this case if $m$ is even, then by Theorem 5.5 only the zero spline interpolates $x(\cdot)$ at the points $jh$, $j \in \mathbb{Z}$, and (5.19) follows at once from (5.15). If $m$ is odd, then, translating $x(\cdot)$ by $h/2$ and using the invariance of the norm with respect to translation, we arrive at the same result.

The inequality (5.20) is obtained by direct use of the Stein inequality [16] if $p = 1$, or of the Hardy-Littlewood-Paley inequality [25] if $p = 2$, or of the Kolmogorov inequality [15] for $p = \infty$, or of the inequality (5.19) for the corresponding value of $p$.

The sharpness of (5.19) and (5.20) is a consequence of more general lower estimates of the $\nu$-widths (see §6.2). The theorem is proved.

The inequality (5.20) was proved for $p = 1$ and $\infty$ in the author's paper [11], and also (including the case $p = 2$) in [26].

The next few results on splines are known in this or that form (see, for example, [27]). We present them here without proof in a convenient form for us.
Suppose that \( h > 0 \), \( t_j = jh \), \( j \in \mathbb{Z} \), \( m \in \mathbb{Z}_+ \), \( N \in \mathbb{N} \), \( N \geq m + 1 \), and \( s \in \mathbb{Z} \). We set
\[
L := \{x(\cdot) \in S^m_{0}(\mathbb{R})| \text{supp } x(\cdot) \subset [t_s, t_{s+N}]\}.
\]
It is easy to see that \( \dim L = N - m \).

**Proposition 5.8.** Let \( 1 \leq p < \infty \). There exist a linear isomorphism \( J : L \cap L^p([t_s, t_{s+N}]) \rightarrow L^p([t_s, t_{s+N}]) \) and constants \( c_i = c_i(m, p) \), \( i = 1, 2 \), such that
\[
\begin{equation}
(5.21)
\end{equation}
\]
\[
\begin{equation}
(5.22)
\end{equation}
\]

**Proposition 5.9.** Suppose that \( 1 \leq p < q \leq \infty \), \( k \in \mathbb{Z}_+ \), and \( k \leq m \). There exists a constant \( c = c(k, m, p, q) \) such that for all \( \xi(\cdot) \in L \)
\[
\begin{equation}
(5.23)
\end{equation}
\]

**Proposition 5.10.** Let \( 1 \leq p \leq \infty \). There exist a projection \( P : L_p([t_s, t_{s+N}]) \rightarrow L \cap L_p([t_s, t_{s+N}]) \) and a constant \( c = c(p) > 0 \) such that \( \|P\| \leq c \). It is important to note that the constants in these propositions are independent of \( s \) and \( N \).

### §6. PROOF OF THE MAIN RESULTS

**6.1. Proof of Theorem 4.1.** Let \( d(W_p(\mathbb{R}), L_q(\mathbb{R})) < \infty \). Then \( d(W_p(\mathbb{R}), L_q(\mathbb{R})) < \infty \) for some \( L \in \text{Lin}_q(L_q(\mathbb{R})) \) such that \( \dim(L_q(\mathbb{R}), \varphi(\cdot)) \leq \nu \).

Let \( \nu > 0 \) and let \( \{\alpha_n\}_{n \in \mathbb{N}} \) be a sequence such that
\[
\begin{equation}
(6.1)
\end{equation}
\]
\[
\begin{equation}
(6.2)
\end{equation}
\]

\( \|x(\cdot)\|_L \) is the space of vectors \( x = (x_1, \ldots, x_n) \), equipped with the norm \( \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \) if \( 1 \leq p < \infty \) and \( \|x\|_{p_0} = \max_{1 \leq i \leq n} |x_i| \).
Estimating the quantity on the left-hand side in (6.1) by the supremum over all functions in $W^r_p(R)$ and the second term on the right-hand side by (6.2), and passing to the supremum over all the indicated $(\cdot)$ in the first term, we get

\[(1 + \varepsilon)d(W^r_p(R), L, L_q(R)) \geq c_1(h(n))^{-1}d(S(n) \cap BL_p(R), Q, L_q(R)) - c_2(h(n))^{\frac{1}{q} - 1/p}.
\]

Let $m(n) = 2[\nu \varphi(\alpha_n)] - [r/2]$. It is easy to see that $m(n) \geq \frac{1}{2} \dim S(n) \geq K_{\varepsilon}(\alpha_n, L, L_q(R))$. Then

\[(1 + \varepsilon)d(W^r_p(R), L, L_q(R)) \geq c_3(h(n))^{-1}d_{m(n)}(S(n) \cap BL_p(R), L_q(R)) - c_2(h(n))^{\frac{1}{q} - 1/p}.
\]

We now estimate the width on the right-hand side in terms of the width of the same set, but with respect to the space $S(n) \cap L_q(R)$ (using Proposition 5.10), and then, using Proposition 5.8, we pass to the widths of the finite-dimensional balls, i.e.,

\[(1 + \varepsilon)d(W^r_p(R), L, L_q(R)) \geq c_4(h(n))^{-1}d_{m(n)}(B_{L_p}(R), L_q(R)) - c_2(h(n))^{\frac{1}{q} - 1/p}.
\]

Since $M(n)/m(n) \to 2$ as $n \to \infty$, it follows that $d_{m(n)}(B_{L_p}^{\alpha(n)}, L_q^{\alpha(n)}) \geq c_4$ for the indicated values of $p$ and $q$ and for sufficiently large $n$, where the constant $c_4 > 0$ depends only on $p$ and $q$. This is a consequence of known asymptotic relations (see, for example, [28]). Thus,

\[(1 + \varepsilon)d(W^r_p(R), L, L_q(R)) \geq (c_4 - c_2)\nu^{-1/p - 1/q}(\alpha_{n}/\varphi(\alpha_n))^{1/q - 1/p}.
\]

Choosing $\varepsilon > 0$ to be sufficiently small from the very beginning, we can assume that $c_4 - c_2 > 0$. Since the quantity on the left-hand side of (6.4) is finite and $r + 1/q - 1/p > 0$ it follows that

\[\liminf_{n \to \infty}(\alpha_n/\varphi(\alpha_n)) < \infty.
\]

We now prove that the last condition suffices for the finiteness of $d_p(W^r_p(R), L_q(R), \varphi(\cdot))$.

Let $1 \leq p \leq q \leq \infty$, $q' = q/(q - 1)$, $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $m \geq r - 1$, and $h > 0$. Then

\[
d(W^r_p(R), S^m(R) \cap L_q(R), L_q(R))
\]

\[
(6.5) \sup_{z(\cdot) \in \tilde{W}^m_R(R, h)} \sup_{x(\cdot) \in W^r_p(R)} \int_{\mathbb{R}} x(t) dW^m(R, h)
\]

where

\[
\tilde{W}^m_{q'}(R, h) = \{z(\cdot) \in \tilde{W}^m_{q'}(R, h) \|z^{(m+1)}(\cdot)\|_{L_{q'}(R)} \leq 1\}.
\]

This follows from the duality relation for the deviation from a subspace and from a description of the annihilator of $S^m(R) \cap L_q(R)$ (see [11]).

Estimating the integral in (6.5) by the Hölder inequality and then using (5.18), we get

\[
d(W^r_p(R), S^m(R) \cap L_q(R), L_q(R)) \leq \sup_{z(\cdot) \in \tilde{W}^m_{q'}(R, h)} \|z^{(m+1)}(\cdot)\|_{L_{q'}(R)}
\]

\[
\leq c h^{-1/q' + 1/p'} = c h^{r - 1/p + 1/q}.
\]
Next, it is easy to verify (using Lemma 2.1) that
\[
\dim(S^m_h(R) \cap L_q(R), L_q(R), \varphi(\cdot)) \leq 2 \lim_{\alpha \to \infty} \inf(\alpha/\varphi(\alpha))h^{-1},
\]
and then (6.6) implies that \(d_r(W^m_p(R), L_q(R), \varphi(\cdot))\) is finite.

If \(\liminf_{\alpha \to \infty}(\alpha/\varphi(\alpha)) > 0\), then from (6.4), (6.6), and the estimate just presented we get the required asymptotic expression for \(d_r(W^m_p(R), L_q(R), \varphi(\cdot))\).

The same estimate and (6.6) show that the spaces \(S^m_{1/\nu}(R) \cap L_q(R), m \geq r - 1\), are asymptotic extremal.

The asymptotic extremality of \(S^m_p(R) \cap L_q(R)\) follows from analogous considerations, where instead of (6.6) it is necessary to use the well-known expression for the asymptotic behavior of the best approximation of Sobolev classes by entire functions of exponential type (see [7]). The theorem is proved.

6.2. Proof of Theorem 4.2. Let us start with the first assertion of the theorem, the one about the value of the mean \(\nu\)-widths.

6.2.1. Lower estimate. Let \(1 < p \leq \infty\). It follows from Theorem 5.3 that
\[
A(p, r)h^{r}BL_p(R) \cap T_h(p, r) \subset W^r_p(R)
\]
for any \(h > 0\). If \(h^{-1} > \nu\), then \(d_r(T_h(p, r) \cap BL_p(R), L_p(R)) = 1\) by Theorem 5.4, and so
\[
\overline{d}_r(W^r_p(R), L_p(R)) \geq A(p, r)h^r.
\]
Since this is true for any \(h^{-1} > \nu\), the required lower estimate for the mean Bernstein \(\nu\)-width is proved. It follows from (3.1) and (3.2) that the same lower estimate is valid for the mean linear \(\nu\)-width and the mean Kolmogorov \(\nu\)-width.

We get a lower estimate for the mean Gel'fand \(\nu\)-width. Suppose that \(Y \hookrightarrow L_p(R)\), \(W^r_p(R) \subset Y\), \(A \subset Y^*\) is a set such that \(L := L(A) \in \mathfrak{L}_p(Y)\), and \(\text{codim}(L, Y) \leq \nu\) (see §1.3). Further, suppose that \(h^{-1} > \nu\), \(\alpha > 0\), \(n = [\alpha/h]\), and \(2[\alpha\nu] \geq r + 1\).

Assume first that \(p = \infty\). Let
\[
S(\alpha) := \{x(\cdot) \in S^m_p(R) | \text{supp} x(\cdot) \subset [-nh, nh]\}.
\]
Then \([-nh, nh] \subset [-\alpha, \alpha]\) and \(\dim S(\alpha) = 2n - r = 2[\alpha/h] - r\). According to (5.12), for \(p = \infty\)
\[
(6.7)
S(\alpha) \cap \frac{K_r h^r}{\pi^r}BL_\infty(R) \subset W^r_\infty(R).
\]
Let \((\alpha_k)_{k \in \mathbb{N}}\) be a sequence such that
\[
\overline{\text{codim}}(L, Y) = \lim_{k \to \infty} (M(\alpha_k, L, Y)/2\alpha_k)
\]
(see 1.3). Since \(\lim_{k \to \infty}(\dim S(\alpha_k)/2\alpha_k) = 1/h\), there exists for \(\varepsilon = (h^{-1} - \nu)/3\) a number \(k_0\) such that for all \(k \geq k_0\)
\[
M(\alpha_k, L, Y) \leq (\nu + \varepsilon)2\alpha_k < (h^{-1} - \varepsilon)2\alpha_k \leq \dim S(\alpha_k).
\]
Hence, \(L(A) \cap Y_{\alpha_k} \cap S(\alpha_k) \neq \emptyset\) for \(k \geq k_0\). Using this and (6.7), we get
\[
\sup \{\|x(\cdot)\|_{L_\infty(R)}|L(A) \cap W^r_\infty(R)\}
\geq \sup \{\|x(\cdot)\|_{L_\infty(R)}|L(A) \cap Y_{\alpha_k} \cap W^r_\infty(R)\}
\geq \sup \left\{\|x(\cdot)\|_{L_\infty(R)}|L(A) \cap Y_{\alpha_k} \cap S(\alpha_k) \cap \frac{K_r h^r}{\pi^r}BL_\infty(R)\right\} = \frac{K_r h^r}{\pi^r}.
\]
Since \(h^{-1} > \nu\) is arbitrary, the required lower estimate is proved for \(p = \infty\).
If $1 < p < \infty$ and $h, \alpha,$ and $n$ are the same as above, then we take $S(\alpha)$ to be the set of $x(\cdot)$ such that $x(nh/\pi) \in T_{2n}(p, r)$ (see Theorem 5.2) and, further, $x^{(i)}(nh) = 0, \ i = 0, 1, \ldots, r - 1.$ As already mentioned (see the proof of Theorem 5.3), $\dim S(\alpha) = 2n - m,$ where $0 \leq m \leq r,$ and the rest of the arguments, with the use of (5.12), are the same as in the case $p = \infty.$

Let $p = 1.$ Here the lower estimate can be proved by a standard device—smoothing of functions in $W_{\nu}^{r-1}(1) := \{x(\cdot) \in L_1(1)| \text{var}(x^{(r-1)}(\cdot)) \leq 1\}$ by the Steklov operator (a similar situation was considered in [11]). We omit the details.

6.2.2. Upper estimate. Let $x(\cdot) \in \mathcal{W}_{\nu}^r(1).$ Theorem 5.5 gives us the existence of a unique interpolation spline $s(x(\cdot), \cdot) \in S_{1/\nu}^{r-1}(1) \cap L_p(1)$ such that
\[ \|x(\cdot) - s(x(\cdot), \cdot)\|_{S_{1/\nu}^{r-1}(1) \cap L_p(1)} \leq A(p, r)\nu^{-r}\|x^{(r)}(\cdot)\|_{L_p(1)} \cdot \]
The correspondence $A: \mathcal{W}_{\nu}^r(1) \rightarrow L_1(1), A(x(\cdot)) := s(x(\cdot), \cdot),$ is obviously linear, and since
\[ \|A(x(\cdot))\|_{L_p(1)} \leq \max(1, A(p, r)\nu^{-r}\|x(\cdot)\|_{L_p(1)} + \|x^{(r)}(\cdot)\|_{L_p(1)} \cdot \]
it follows that $A \in \mathcal{L}(\mathcal{W}_{\nu}^r(1), L_p(1)).$

Since $\text{Im} A \subseteq S_{1/\nu}^{r-1}(1) \cap L_p(1),$ we have $\overline{\text{dim}}(\text{Im} A, L_p(1)) \leq \nu,$ and the upper estimate for the mean linear $\nu$-width is thereby proved. By virtue of (3.1) and (3.2) this estimate is proved in the same way for the mean Kolmogorov and Bernstein $\nu$-widths.

The upper estimate for the mean Gel'fand $\nu$-width follows from the first inequality in Theorem 5.7 for $h = 1/\nu$ and $m = r.$

6.2.3. The equalities $A(1, r) = A(\infty, r) = \pi^{-r}K_r$ follow from known extremal properties of Euler splines (see, for example, [10]), which are solutions of the problem (4.1) for $p = 1$ and $\infty.$

Let $p = 2.$ In this case (5.3) is equivalent to the equation
\[ x_n(t) = (-1)^r\lambda_{2n}^{(2r)}(t) \cdot \]
The pair $(\lambda_n, x_n(\cdot)), \lambda_n = x^n$ and $x_n(t) = n^{-r}n^{-1/2}\sin nt,$ satisfies this equation and the conditions (5.4), as is easily verified. Using the expression for $\lambda_n$ and the definition of $A(2, r),$ we get
\[ A(2, r) = 2^{-r}\|x(\cdot)\|_{L_2([0, 1])} = 2^{-r}\lambda_n^{(2r)}(2n/n)^r = 2^{-r}n^{-r}(2n/n)^r = \pi^{-r} \cdot \]

If $r = 1,$ then a solution of (4.1) can be found explicitly (see [12]), and a direct computation leads to the corresponding expression for $A(p, 1), 1 < p < \infty.$

6.2.4. We proceed to a description of extremal spaces and operators. The extremality of the pair $(\mathcal{W}_{\nu}^r(1), A)$ for $A_{\nu}(\mathcal{W}_{\nu}^r(1), L_p(1)), 1 \leq p \leq \infty,$ where $A(x(\cdot))$ is an interpolation spline, was proved above (in getting an upper estimate for this quantity). This yields, in particular, the extremality of the space $S_{1/\nu}^{r-1}(1) \cap L_p(1)$ for $A_{\nu}(W_{\nu}^r(1), L_p(1)), 1 \leq p \leq \infty.$

Let $p = 2$ and let $H$ be the operator defined in the theorem. Clearly this is a continuous linear operator, and $\text{Im} H \subseteq \mathcal{S}_{2\nu}(1) \cap L_2(1)$. Then $\overline{\text{dim}}(\text{Im} H, L_2(1)) \leq \nu,$ by Lemma 2.1. If we now use Plancherel's theorem, we can verify easily that
\[ \|x(\cdot) - Hx(\cdot)\|_{L_2(1)} \leq (\pi\nu)^{-r}, \]
i.e., $(\mathcal{W}_{\nu}^r(1), H)$ is an extremal pair for $A_{\nu}(W_{\nu}^r(1), L_2(1)).$
Let \( p = q \) in (6.5). Estimating its right-hand side by the Hölder inequality (see the first inequality in (6.6)), and then using (5.20) with \( m + 1 \) instead of \( m \) and with \( k = m - r + 1 \) and assuming that \( p = 1, 2, \) or \( \infty \), we get that

\[
(6.8) \quad d(W^p_{n}(\mathbb{R}), S^m_{p}(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R})) \leq A(p', r) h'.
\]

The quantity \( A(p, r), 1 \leq p \leq \infty \), is the value of the problem (4.1) up to a factor, and thus \( A(p, r) = A(p', r), p' = p/(p-1) \) (see [13]). Consequently, (6.8) means that for \( p = 1, 2, \) or \( \infty \) the space \( S^m_{p}(\mathbb{R}) \cap L_p(\mathbb{R}) \) is extremal for \( d(W^p_{n}(\mathbb{R}), L_p(\mathbb{R})) \) when \( m > r - 1 \).

The extremality of \( S^m_{p}(\mathbb{R}) \cap L_p(\mathbb{R}) \) for \( d(W^p_{n}(\mathbb{R}), L_p(\mathbb{R})) \) when \( p = 1 \) or \( \infty \) is a consequence of a theorem of M. Krein [29] and Lemma 2.1, and the extremality of \( S^m_{p}(\mathbb{R}) \cap L_2(\mathbb{R}) \) follows from the extremality of the pair \((\mathcal{H}_p^r(\mathbb{R}), H)\) for \( \lambda(p(W^p_{n}(\mathbb{R}), L_2(\mathbb{R}))) \).

It follows from (5.19) for \( m = r \) that the pair \((\mathcal{H}_p^r(\mathbb{R}), L)\), where

\[
L = \{ x(\cdot) \mathcal{H}_p^r(\mathbb{R}) | x(j/\nu) = 0, j \in \mathbb{Z} \},
\]

is extremal for \( d(W^p_{n}(\mathbb{R}), L_p(\mathbb{R})) \) when \( 1 \leq p \leq \infty \). The theorem is proved.

We remark that the value of \( d(W^p_{n}(\mathbb{R}), S^m_{p}(\mathbb{R}) \cap L_p(\mathbb{R}), L_p(\mathbb{R})) \) for \( p = 1 \) and \( \infty \) and for \( m > r - 1 \) was found earlier in the author’s paper [11] (see also [17]) and, including the case \( p = 2 \), in [26].

§7. THE RECOVERY PROBLEM

The quantities introduced above are closely connected with the following problem of optimal recovery of functions in the class \( \mathcal{H}_p^r(\mathbb{R}) \).

Suppose that \( 1 \leq p \leq q \leq \infty, r \in \mathbb{N}, J \) is some collection of pairs \((Y, I, \) where \( Y \) is a linear space, \( I: \mathcal{H}_p^r(\mathbb{R}) \to Y \) is a linear operator, and \( \mathcal{F}(Y, I) \) is the set of all mappings from \( \text{Im} I \) to \( L_q(\mathbb{R}) \). The quantity

\[
E(W^p_{n}(\mathbb{R}, J, L_q(\mathbb{R})) := \inf_{(Y, I) \in J} \inf_{\mathcal{F}(Y, I)} \sup_{x(\cdot) \in \mathcal{H}_p^r(\mathbb{R})} \| x(\cdot) - (F \circ I)x(\cdot) \| L_q(\mathbb{R})
\]

is called the exact error of recovery of elements in \( \mathcal{H}_p^r(\mathbb{R}) \) from the information \( J \) in \( L_q(\mathbb{R}) \).

If

\[
E(W^p_{n}(\mathbb{R}, J, L_q(\mathbb{R}))) = \sup_{x(\cdot) \in \mathcal{H}_p^r(\mathbb{R})} \| x(\cdot) - (\hat{F} \circ \hat{I})x(\cdot) \| L_q(\mathbb{R})
\]

for some \((\hat{Y}, \hat{I}) \in J \) and \( \hat{F} \in \mathcal{F}(\hat{Y}, \hat{I}), \) then we say that \((\hat{Y}, \hat{I}) \) is optimal information, and \( \hat{F} \) is an optimal method of recovery.

**Theorem 7.1.** Suppose that \( 1 \leq p \leq \infty, r \in \mathbb{N}, \nu > 0, \) and \( J_\nu \) is the collection of all pairs \((Y, I)\) such that \( Y \) is a linear space and \( I := \mathcal{H}_p^r(\mathbb{R}) \to Y \) is a linear operator with

\[
\text{codim} (\text{Ker} I, \mathcal{H}_p^r(\mathbb{R})) \leq \nu.
\]

Then

\[
E(W^p_{n}(\mathbb{R}, J_\nu, L_p(\mathbb{R}))) = A(p, r)\nu^{-r},
\]

where \( A(p, r) \) is defined in Theorem 4.2.

The information \((\hat{Y}, \hat{I}), \) where \( \hat{Y} \) is the linear space \( l^\infty(\mathbb{Z}) \) of all two-sided bounded sequences and \( \hat{I} \) assigns to an \( x(\cdot) \in \mathcal{H}_p^r(\mathbb{R}) \) the sequence

\[
\{ x(j/\nu + (1 - (-1)^j)/4\nu) \}_{j \in \mathbb{Z}},
\]

is optimal.
The mapping $\hat{F}$ that associates with $\{x(j/\nu + (1 - (-1)^j)/4\nu)\}_{j \in \mathbb{Z}} \in \text{Im} \hat{f}$ the unique spline $s(\cdot) \in S_{1/\nu}^{-1}(R) \cap L_p(R)$ with

$$s(j/\nu + (1 - (-1)^j)/4\nu) = x(j/\nu + (1 - (-1)^j)/4\nu), \quad j \in \mathbb{Z},$$

is an optimal method of recovery.

Proof. It is not hard to verify that

$$E(W_p'(R), J_\nu, L_p(R)) \geq d^{\nu}(W_p'(R), L_p(R)),$$

and thus the lower estimate follows from Theorem 4.2.

Further, if $A := \hat{F} \circ \hat{f}$, then, again by Theorem 4.2, the pair $(W_p'(R), A)$ is extremal for $\lambda_p(W_p'(R), L_p(R))$. This proves the upper estimate, as well as the optimality of the information $(\hat{Y}, \hat{I})$ and of the method $\hat{F}$ of recovery. The theorem is proved.

Denote by $\mathcal{J}_\nu$ the subset of $J_\nu$ formed by the pairs $(Y, I)$ for which $Y = l^{\infty}(\mathbb{Z})$ and $I_X(\cdot) = I_{\xi}X(\cdot) = \{x(t_j)\}_{j \in \mathbb{Z}}$, where $\xi = \{t_j\}_{j \in \mathbb{Z}} \in \Xi$. Then it is clear that

$$(Y, I_{\xi}) \in \mathcal{J}_\nu \text{ if and only if } \liminf_{n \to \infty} N(\alpha, \xi) \geq \nu,$$

where, we recall, $N(\alpha, \xi)$ is the number of points of the sequence $\xi$ that belong to $(-\alpha, \alpha)$.

Note that $\mathcal{J}_\nu$ does not exhaust $J_\nu$. Indeed, if, for example, $r \geq 2$, $t_j = jr/\nu$, $j \in \mathbb{Z}$, and $I_X(\cdot) = \{x^{(i)}(t_j)\}_{j \in \mathbb{Z}}, \quad i = 0, 1, \ldots, r - 1$, then

$$\text{codim}(\text{Ker} I, W_p'(R)) = \nu$$

by Lemma 2.2. It follows from Theorem 7.1 that

$$E(W_p'(R), \mathcal{J}_\nu, L_p(R)) = A(p, r)^{-r},$$

with the same optimal information and the same optimal method of recovery.

This result was proved by Sun Yongsheng [24] for $p = \infty$, by Li Chun [22] for $p = 1$, and by Sun Yongsheng and Li Chun [23] for $p = 2$, and it was announced by Li Chun for all the remaining $p$ (see [30]).

Bibliography


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