For subspaces of $L_p(\mathbb{R})$ the author introduces the concept of $\varphi$-average dimension. For $f \in L_p(\mathbb{R})$, let $P_\alpha f(t) = f(t)I_\alpha(t)$, where $I_\alpha(t)$ is the indicator function of $[-\alpha, \alpha]$. Let $L$ be a subspace of $L_p(\mathbb{R})$ and let

$$K_\varepsilon(\alpha, L, L_p(\mathbb{R})) := \min\{n \in \mathbb{Z}_+: d_n(L \cap BL_p), L_p(\mathbb{R})) < \varepsilon\},$$

where $BL_p$ is the unit ball of $L_p(\mathbb{R})$ and $d_n$ are Kolmogorov $n$-widths. Let $\varphi$ be a nondecreasing positive function on $(0, \infty)$, with $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$. For $1 \leq p \leq \infty$, the quantity

$$\lim_{\varepsilon \to 0} \liminf_{\alpha \to \infty} K_\varepsilon(\alpha, L, L_p(\mathbb{R}))/\varphi(\alpha)$$

is called the $\varphi$-average dimension of $L$ in $L_p(\mathbb{R})$. The $\varphi$-average codimension is defined similarly. The $\varphi$-average widths of the set $K$ in $L_p(\mathbb{R})$ are then defined in the standard way using $\varphi$-average dimensions (codimensions) of approximating spaces instead of conventional dimensions (codimensions). The author finds the conditions, in terms of $\varphi$, for the finiteness of the $\varphi$-average Kolmogorov widths and determines various $\varphi$-average widths for the Sobolev classes.

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AVERAGE DIMENSION AND WIDTHS OF FUNCTION CLASSES ON THE LINE

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In this note we introduce the concept of average dimension for a sufficiently broad class of infinite-dimensional subspaces of $L_p(R)$ and define corresponding analogues of the linear widths, Kolmogorov widths, and Bernstein widths. We present precise values of them for the Sobolev function classes $W^r_p(R)$ in $L_p(R)$ for all $1 \leq p \leq \infty$. Sharp results on approximation by splines and entire functions are given a natural interpretation in these terms: they turn out to be extremal spaces.

1. Average dimension. Let $(X, \| \cdot \|)$ be a normed space. We use the following notation: $B_X$ is the unit ball in $X$, $\text{Lin}(X)$ is the collection of all linear subspaces of $X$, and $\text{Lin}_n(X) := \{ L \in \text{Lin}(X) \mid \dim L \leq n \}$, where $n \in \mathbb{Z}^+$.

If $A, C \subset X$, then

$$d(C, A, X) := \sup_{x \in C, y \in A} \| x - y \|$$

is the best approximation of $C$ by the set $A$.

The notation $L_p(G)$ has the standard sense, where $1 \leq p \leq \infty$ and $G$ is a closed interval or the line $R$.

The restriction of a set $A \subset L_p(R)$ to the interval $[-T, T]$, $T > 0$, is denoted by $A_T$ (in particular, $x_T(\cdot) := \text{restriction of } x(\cdot) \in L_p(R)$ to $[-T, T]$).

Let $\text{Lin}_n(L_p(R))$, $1 \leq p \leq \infty$, denote the collection of subspaces $L \in \text{Lin}(L_p(R))$ such that $x_T(\cdot) \in L$ is precompact in $L_p([-T, T])$ for any $T > 0$.

If $L \in \text{Lin}_n(L_p(R))$, $T > 0$, and $\varepsilon > 0$, then there exist an $n \in \mathbb{Z}^+$ and an $M \in \text{Lin}_n(L_p([-T, T]))$ such that

$$d((L \cap BL_p(R))_T, M, L_p([-T, T])) < \varepsilon.$$

Let

$$\mathcal{D}_\varepsilon(T; L; L_p(R)) := \min \{ n \in \mathbb{Z}^+ \mid \exists M \in \text{Lin}_n(L_p([-T, T])) : d((L \cap BL_p(R))_T, M, L_p([-T, T])) < \varepsilon \}.$$

This function is nondecreasing with respect to $T$ and nonincreasing with respect to $\varepsilon$.

DEFINITION 1. Let $L \in \text{Lin}_n(L_p(R))$, $1 \leq p \leq \infty$. The quantity

$$\dim(L; L_p(R)) := \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\mathcal{D}_\varepsilon(T; L; L_p(R))}{2T}$$

is called the average dimension of $L$ in $L_p(R)$.

This definition is an insignificant modification of the corresponding concept introduced earlier by Tikhomirov [1]. (It is possible to give a more general definition— the $\varphi$-average dimension—when an arbitrary nondecreasing function $\varphi(\cdot)$ appears...)

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instead of $2T$ in (1) (see [2]), but, as follows from [2], in the case of compatible metrics it suffices to confine oneself to the definition given.

Let $m \in \mathbb{Z}_+$ and $h > 0$. Denote by $S_h^m$ the collection of polynomial splines on $\mathbb{R}$ of order $m$ and defect 1 with nodes at the points $jh$, $j \in \mathbb{Z}$. It is clear that $S_h^m \cap L_p(\mathbb{R}) \in \text{Lin}_{\varepsilon}(L_p(\mathbb{R}))$, $1 \leq p \leq \infty$.

Let $\sigma > 0$ and $1 \leq p \leq \infty$. Denote by $B_{\sigma,p}$ the restriction to $\mathbb{R}$ of the set of all entire functions of exponential type $\sigma$ belonging to $L_p(\mathbb{R})$. It follows from the Bernstein-Nikol’skii skit theorem ([3], Theorem 3.3.6) that $B_{\sigma,p} \in \text{Lin}_c(L_p(\mathbb{R}))$, $1 \leq p \leq \infty$.

**Lemma 1.** Suppose that $1 \leq p \leq \infty$, $m \in \mathbb{Z}_+$, $h > 0$, and $\sigma > 0$. Then:

(a) $\overline{\text{dim}}(S_h^m \cap L_p(\mathbb{R})); L_p(\mathbb{R})) = h^{-1}$;

(b) $\overline{\text{dim}}(B_{\sigma,p} ; L_p(\mathbb{R})) = \sigma / \pi$.

For $p = \infty$ formula (b) was actually obtained in [4], and for $1 < p < \infty$ in [5].

2. Definitions of the widths. Suppose that $1 \leq p \leq \infty$, $W$ is a centrally symmetric subset of $L_p(\mathbb{R})$, and $N > 0$.

**Definition 2.** The average Kolmogorov $N$-width of $W$ in $L_p(\mathbb{R})$ is defined to be

$$
\overline{d}_N(W, L_p(\mathbb{R})) := \inf \sup L \inf \|x(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})},
$$

where the infimum is over all $L \in \text{Lin}_c(L_p(\mathbb{R}))$ such that $\overline{\text{dim}}(L; L_p(\mathbb{R})) \leq N$.

A space on which this infimum is attained is said to be extremal.

The average linear $N$-width of $W$ in $L_p(\mathbb{R})$ is defined to be

$$
\overline{\ell}_N(W, L_p(\mathbb{R})) := \inf \sup (X, X) \in W \|x(\cdot) - \Lambda x(\cdot)\|_{L_p(\mathbb{R})},
$$

where the infimum is over all pairs $(X, \Lambda)$ such that $X$ is a normed space continuously imbedded in $L_p(\mathbb{R})$, $W \subseteq X$ and $\Lambda: X \rightarrow L_p(\mathbb{R})$ is a continuous linear operator such that $\text{Im} \Lambda \in \text{Lin}_c(L_p(\mathbb{R}))$ and $\overline{\text{dim}}(\text{Im} \Lambda; L_p(\mathbb{R})) \leq N$.

A pair on which this infimum is attained is said to be extremal.

The average Bernstein $N$-width of $W$ in $L_p(\mathbb{R})$ is defined to be

$$
\overline{b}_N(W, L_p(\mathbb{R})) := \sup \sup \{\lambda > 0 \mid L \cap \lambda BL_p(\mathbb{R}) \subseteq W\},
$$

where the infimum is over all $L \in \text{Lin}_c(L_p(\mathbb{R}))$ such that $\overline{\text{dim}}(L; L_p(\mathbb{R})) > N$ and $\overline{d}_N(L \cap BL_p(\mathbb{R}), L_p(\mathbb{R})) = 1$.

The last condition means that only those spaces are considered for which the analogue of the Tikhomirov theorem on the width of a ball is valid. It is shown in [2] that the spaces $S_h^m \cap L_p(\mathbb{R})$ and $B_{\sigma,p}$ satisfy this requirement.

3. Formulation of the main result. Suppose that $1 \leq p \leq \infty$ and $r \in \mathbb{N}$. Denote by $Z'_r(\mathbb{R})$ the collection of functions $x(\cdot) \in L_p(\mathbb{R})$ such that the $(r-1)$st derivative $x^{(r-1)}(\cdot)$ is locally absolutely continuous, and $x^{(r)}(\cdot) \in L_p(\mathbb{R})$. This is a Banach space with the norm $\|x(\cdot)\|_{L_p(\mathbb{R})} + \|x^{(r)}(\cdot)\|_{L_p(\mathbb{R})}$.

Let $W'_r(\mathbb{R}) := \{x(\cdot) \in Z'_r(\mathbb{R}) \mid \|x^{(r)}(\cdot)\|_{L_p(\mathbb{R})} \leq 1\}$.

**Theorem 1.** Suppose that $1 \leq p \leq \infty$, $r \in \mathbb{N}$, and $N > 0$. Then

$$
\overline{b}_N(W'_r(\mathbb{R}) , L_p(\mathbb{R})) = \overline{d}_N(W'_r(\mathbb{R}) , L_p(\mathbb{R})) = \overline{\ell}_N(W'_r(\mathbb{R}) , L_p(\mathbb{R})) = A(p, r)N^{-r},
$$
where $A(p, r) = 2^{-\gamma} \|\tilde{x}(\cdot)\|_{L_p(0, 1)}$, and $\tilde{x}(\cdot)$ is the unique solution of the extremal problem

$$
\|x(\cdot)\|_{L_p(0, 1)} = \sup_{x_i(\cdot)} \|x_i^{(r)}(\cdot)\|_{L_p(0, 1)} \leq 1,
$$

$x_i^{(r)}((1 - (-1)^r)/2) = 0, \quad 0 \leq i \leq r - 1$.

In particular, $A(1, r) = A(\infty, r) = \pi^{-r} K_r$, where

$$K_r := \frac{4}{\pi} \sum_{j=0}^{\infty} (-1)^{j(r+1)}(2j + 1)^{-r+1},$$

$A(2, r) = \pi^{-r}, \quad A(p, 1) = p \sin \frac{\pi}{p} / 2\pi(p - 1)^{1/p}$ if $1 < p < \infty$.

The pair $(\mathscr{B}^r_p(R), \Lambda)$, where $\Lambda x(\cdot) = S_{1/N}^{-1} \cap L_p(R)$ and $\Lambda x(k/N + (1 - (-1)^r)/4N)$ for $k \in \mathbb{Z}$, is extremal for $\bar{\Lambda}_p(W^r_p(R), L_p(R))$ when $1 \leq p \leq \infty$. For $p = 2$, the pair $(\mathscr{B}^r_p(R), H)$ is also extremal, where $H$ is determined from the condition $FHx(\cdot) = \chi_{\pi N}(\cdot)Fx(\cdot)$, $F$ being the Fourier transformation acting in $L_2(R)$, and $\chi_{\pi N}(\cdot)$ the characteristic function of the interval $[-\pi N, \pi N]$.

The spaces $S_{1/N}^m \cap L_p(R)$ for all $m \geq r - 1$ and the spaces $B_{\pi N, p}$ are extremal for $d_B(W^p_p(R), L_p(R))$ when $p = 1, 2, or \infty$.

The following two results are used essentially in the proof of Theorem 1 and are of independent interest.

**Theorem 2.** Suppose that $1 < p \leq \infty$, $r \in \mathbb{N}$, and $N > 0$. There exists a subspace $L \subset \mathscr{B}_p^r(R)$ such that $L \in \text{Lin}(L_p(R))$, $\dim(L; L_p(R)) = N$ and for all $x(\cdot) \in L$

$$(2) \quad \|x^{(r)}(\cdot)\|_{L_p(R)} \leq \frac{N^r}{A(p, r)} \|x(\cdot)\|_{L_p(R)},$$

where $A(p, r)$ is defined in Theorem 1.

For $1 < p < \infty$ this inequality (of Bernstein type) is a consequence of its periodic analogue

$$(3) \quad \|x^{(r)}(\cdot)\|_{L_p([-\pi, \pi])} \leq \frac{n^r}{\pi A(p, r)} \|x(\cdot)\|_{L_p([-\pi, \pi])},$$

which is valid (for each $n \in \mathbb{N}$) for some $2n$-dimensional subspace of $L_p([-\pi, \pi])$.

The proof of (3) is based on results of Buslaev and Tikhomirov [6].

For $p = \infty$ inequality (2) is another way of writing a well-known inequality for functions in $S_{1/N}^r \cap L_\infty(R)$ (see [7]).

**Theorem 3.** Suppose that $1 \leq p < \infty$, $r \in \mathbb{N}$, and $h > 0$. For each $x(\cdot) \in W^r_p(R)$ there exists a unique spline $s(x(\cdot), \cdot) \in S_{h}^{r-1} \cap L_p(R)$ such that

$$s(x(\cdot), kh + (1 - (-1)^r)h/4) = x(kh + (1 - (-1)^r)h/4), \quad k \in \mathbb{Z},$$

and, further,

$$(4) \quad \|x(\cdot) - s(x(\cdot), \cdot)\|_{L_p(R)} \leq A(p, r)h^r,$$

where $A(p, r)$ is defined in Theorem 1.

*(1) If $p = 1$, then the constraint $\|x^{(r)}(\cdot)\|_{L_p(0, 1)} \leq 1$ must be replaced by $\text{var}(x^{(r-1)}(\cdot)) \leq 1$. *
The proof of this result is also based on the corresponding periodic analogue, which was established in [8] for $p = \infty$, in [9] for $p = 1$, and in [6] for $1 < p < \infty$. Relation (4) was proved earlier for $h = 1$ and $p = 1, 2,$ and $\infty$, respectively, by Li Chun (preprint), Sun Yong Sheng and Li Chun (preprint), and in [10].

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