The differential equation $y'' - y^p = \pm (p > 0)$. (English summary)


From the introduction: “The differential equation $y' = y$ is solved by $y = Ae^x$. Many differential equations (not necessarily of the first order) close to $y' = y$ in some sense can be solved by expanding in a series of the type (1) $y = Ae^x + a_1(x)A^{1-p}e^{(1-p)x} + a_2(x)A^{1-2p}e^{(1-2p)x} + \cdots$, where the coefficients $a_1, a_2, \cdots$ (and the choice of $p$) depend on the situation at hand. Up to our knowledge this method was, for the purpose of numerical integration, first used, more or less systematically, by E. Meissel in the 1870’s.

“In another paper Meissel discovered for the differential equation $dy/dx = +\sqrt{y^2 - \cos 2x}$ a remarkable property of its solutions: there are algebraic relations between the function values at the points $\pi/4$, $3\pi/4$, $5\pi/4$, $\cdots$, although the solutions themselves are not algebraic functions. In this case the coefficients are periodic functions of period $\pi$ and the statement about algebraic dependence is connected with the fact that $a_2(\pi/4) = a_3(\pi/4) = \cdots = 0$. As far as we know, the proof of the theorem was never published.

“In this paper we apply Meissel’s method to the differential equation (2) $y'' - y^p = 1$ ($p > 0$), which arose in connection with the determination of a $K$-functional. Because of translation invariance the coefficients $a_1, a_2, \cdots$ in (1) are in this case constants. The properties of the expansion (1) in this case are investigated in Sections 1–3. We consider mostly only real solutions $y$ such that $y > 0$, $y' > 0$.

“We are primarily interested in the case when the parameter range is $p > 0$ (sometimes we have to make the restriction $p > 1$; see Section 5). But also the case $p = -1$ and, generally speaking, $p = -1/n$ ($n \in \mathbb{N}$) enters at least implicitly in the discussion (see Section 6) and in that case the equation can be integrated in terms of elementary functions (inverses of algebro-logarithmic functions). For instance, for $p = -1$ a solution is given by the functional equation $x = \log y + y$ and so forth. Furthermore, such a solution exists also for, e.g., $p = -1/2$, and then the corresponding relation reads $x = \log y - 2/(y^2 - 1)$.

“In Section 4 we treat equation (2) by the method of separation of variables.

“In the same way as (2) we can treat the companion equation (2') $y'' - y^p = -1$ ($p > 0$). We are then dealing with a generalization of the hyperbolic cosine. The equation (2') and the generalized hyperbolic cosine will be treated in some detail only in Section 5.

“We consider also (Section 6) power series expansions for our generalized hyperbolic sine and cosines. These are useful in the numerical computations of these functions close to the origin (Section 7).

“Finally, in Section 8 we return to the problem that instigated this research—connected with the determination of the best constant in a certain inequality.”
Editor’s remark: Apparently, the equation in the title should read $y'^p - y^p = \pm 1 \ (p > 0).$

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