Sturm–Liouville Type Problems for the $p$-Laplacian under Asymptotic Non-resonance Conditions

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1. INTRODUCTION AND MAIN RESULTS.

For the differential operator $L_p^a u = r^{-\frac{1}{p}}(r^{p} u^{(p-1)})'$ with $\alpha \geq 0$ and $p > 1$ we study the equation

$$L_p^a u + f(r, u) = 0 \quad \text{in } I = [a, b], \quad 0 \leq a < b < \infty$$

with Sturm–Liouville type boundary conditions

$$\gamma_1 u^{(p-1)}(a) + \gamma_2 (r^{p} u^{(p-1)})(a) = u_0,$$  
(2)

$$\gamma_3 u^{(p-1)}(b) + \gamma_4 (r^{p} u^{(p-1)})(b) = u_1,$$  
(3)

where $\gamma_i, u_j \in \mathbb{R}$ ($i = 1, \ldots, 4; j = 0, 1$), $\gamma_1^2 + \gamma_2^2 > 0$, $\gamma_3^2 + \gamma_4^2 > 0$ and $u^{(q)} = |u|^{q-1} u$ is the odd power-function. We distinguish two cases: the regular case (R) where $a > 0$ or $a = 0$ and $0 \leq \alpha < p - 1$, and the singular case (S) defined by $a = 0$, $\alpha \geq p - 1$. In the singular case the boundary condition at $0$ is

$$u'(0) = 0.$$  
(2)

We write the boundary condition (2), (3) as $(Bu)(a) = u_0$, $(Bu)(b) = u_1$ or simply as $Bu = (u_0, u_1)$. With (1), (2), (3) we associate the eigenvalue problem

$$L_p^a u + (q(r) + \lambda s(r)) u^{(p-1)} = 0 \quad \text{in } I,$$  
(4)

$$Bu = (0, 0),$$  
(5)

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where \( q, s \in L^\infty(I) \) and \( \text{ess inf}_I s > 0 \). The value \( \lambda \) is called an eigenvalue and \( u_* \not\equiv 0 \) an eigenfunction if the pair \((\lambda, u_*)\) satisfies (4), (5).

The following Theorem first appeared in Walter [15]. It generalizes a classical and well known theorem for \( p = 2 \) to general \( p > 1 \). For the radial \( p \)-Laplacian the existence of eigenvalues was shown by DelPino, Manásevich [5] for the special case \( q \equiv 0, s \equiv 1 \). The results of our paper are based on a Prüfer-type transformation, which is well known for \( p = 2 \) and new for general \( p > 1 \), cf. Section 3. It transforms the second-order equation (4) into two first-order equations in the phase-plane, to which the theory of differential inequalities applies.

**Theorem 1.** The eigenvalue problem (4), (5) has a countable number of simple eigenvalues \( \lambda_1 < \lambda_2 < \cdots \), \( \lim_{n \to \infty} \lambda_n = \infty \), and no other eigenvalues. The corresponding eigenfunction \( u_n \) has \( n - 1 \) simple zeroes in \( I_0 = (a, b) \). Between \( a \) and the first zero of \( u_n \), between any two consecutive zeroes of \( u_n \) and also between the last zero of \( u_n \) and \( b \) there is exactly one zero of \( u_{n+1} \).

We also consider the following generalization of (4), (5)

\[
L^* u + q(r) u^{\gamma p - 1} + s(r) (\mu u_*)^{\gamma p - 1} - v (u - v)^{\gamma p - 1} = 0 \quad \text{in } I, \\
Bu = (0, 0),
\]

where \( u^* (x) = \max(u(x), 0) \) and \( u = u^* - u^- \). The pair \((\mu, v)\) is called a Fučík-eigenvalue and \( u_{\mu_*}, \not\equiv 0 \) a Fučík-eigenfunction if \((\mu, v, u_{\mu_*})\) solves (6), (7). The set \( \sigma \) of all Fučík-eigenvalues \((\mu, v)\) is called the Fučík-spectrum. The importance of the Fučík-spectrum was first discovered by Fučík [8] and Dancer [3] in the case \( p = 2 \). Important results on the Fučík-spectrum have been achieved for general, linear self-adjoint operators by Schechter [14] and for self-adjoint second-order operators on general domains by deFigueiredo, Gossez [4]. For the radially symmetric Laplace-operator with \( q \equiv 0, s \equiv 1 \) we refer to Arias, Campos [1]. Results on the Fučík-spectrum for \( p > 1 \) and \( \alpha = 0 \) can be found in Drábek [6].

With the help of the Prüfer-transformation we obtain the following complete global description of the Fučík-spectrum in the general setting (6), (7). Notice that a function \( u \) is called initially positive or negative at \( a \) if \( u(a + \epsilon) > 0 \), i.e. \( u(r) > 0 \) in \((a, a + \epsilon)\) for some \( \epsilon > 0 \), or \( u(a + ) < 0 \), resp.

**Theorem 2.** The Fučík-spectrum \( \sigma \) is closed in the \( \mu \nu \)-plane \( \mathbb{R}^2 \). It takes the form \( \sigma = \sigma^+ \cup \sigma^- \), where \((\mu, v) \in \sigma^+ \) if and only if \((v, \mu) \in \sigma^- \). The sets \( \sigma^+, \sigma^- \) are closed and consist of Fučík-eigenvalues with initially positive, initially negative Fučík-eigenfunctions. Furthermore \( \sigma \cap \{(\lambda, \lambda) : \lambda \in \mathbb{R}\} = \{(\lambda_k, \lambda_k) : k \in \mathbb{N}\} \subseteq \sigma^+ \cap \sigma^- \). The Fučík-eigenfunctions corresponding to the connected components \( \sigma^+_k, \sigma^-_k \) \( (k \in \mathbb{N}) \) of \( \sigma^+, \sigma^- \) have exactly \( k - 1 \) zeroes.
in \( I^0 \). The first component \( \sigma^+_k \) consists of \( \{ \lambda_i \} \times \mathbb{R} \), while for \( k \geq 2 \), \( \sigma^-_k \) is a \( C^1 \)-curve \( (\mu, v(\mu)) \) with \( v'((\mu)) < 0 \) and with the following asymptotics \( (\text{the relations for } \sigma^-_k \text{ follow by symmetry}) \):

\begin{align*}
(\text{S}) & \quad \text{Case } a = 0 \text{ and } x \geq p - 1:
& \quad k = 2i: \quad v(\infty) = \lambda^+_i, \quad v(\lambda^{+}_{i} +) = \infty \\
& \quad k = 2i + 1: \quad v(\infty) = \lambda^-_i, \quad v(\lambda_{i+1}^- +) = \infty
\end{align*}

\begin{align*}
(\text{R}) & \quad \text{Case } a > 0 \text{ or } 0 \leq x < p - 1:
& \quad k = 2i: \quad v(\infty) = \lambda^a_i, \quad v(\lambda^a_{i} +) = \infty \\
& \quad k = 2i + 1: \quad v(\infty) = \lambda^{ab}_i, \quad v(\lambda_{i+1}^{ab} +) = \infty
\end{align*}

where \( \lambda^a_i, \lambda^b_i, \lambda^{ab}_i \) are the eigenvalues to (4) with the boundary conditions

\begin{align*}
(5^a) & \quad u(a) = 0, \quad u(b) = 0 \\
(5^b) & \quad u(a) = 0, \quad (Bu)(b) = 0 \\
(5^{ab}) & \quad (Bu)(a) = 0, \quad u(b) = 0
\end{align*}

**Remark.** (1) We do not know the characterization of \( \sigma^+ \cap \sigma^- \). There are cases (cf. Arias, Campos [1]) where \( \sigma^+_k = \sigma^-_k \). We conjecture that either \( \sigma^+_k = \sigma^-_k \) or \( \sigma^+_k \cap \sigma^-_k = \{ (\lambda_k, \lambda_k) \} \).

(2) The regular case (R) and the singular case (S) have a noticeable difference in the asymptotic behaviour because in general, i.e., for \( \gamma_2 \neq 0 \), we have \( \lambda_i < \lambda^a_i \) and \( \lambda_i < \lambda^{ab}_i \). For \( p = 2 \) and \( x = N - 1 \) this was noticed in [4] as a difference between the one-dimensional case \( N = 1 \) and the higher-dimensional case \( N \geq 2 \). For \( p > 2 \) the difference occurs at the dividing dimension given by the smallest integer \( \leq p \).

With the help of the Fučik-spectrum we can now state an existence theorem for the non-homogeneous problem (1), (2), (3). Similar theorems for \( p = 2 \) were obtained by Fučik [8], Dancer [3], DeFigueiredo, Gossez [4] and in the special cases \( x = 0, p \geq 2 \) by Boccardo et al. [2] and in the regular case (R) for all \( p > 1 \) by Huang, Metzen [9]. Unlike the previous results, which make use of degree-theory, our results are obtained by means of the Prüfer-transformation. They are valid in a general setting.

**Theorem 3.** Let \( f \) be continuous in \( I \times \mathbb{R} \) and satisfy a uniqueness condition for the family of initial value problems (1) with initial values \( u(0) = \tau \),
2. GENERALIZED SINE-FUNCTIONS

We consider the solution $S_\rho(\phi) = \sin_\rho(\phi)$ of

$$u'' + \frac{u'}{p} - \frac{1}{p - 1} = 1, \quad u(0) = 0, \quad u'(0) = 1$$

as long as $u$ is increasing, i.e., on the interval $[0, \pi_p/2]$ with

$$\frac{\pi_p}{2} = \int_0^{\pi_p/(p - 1)} \frac{dt}{(1 - t^p/(p - 1))^{1/p}} = \frac{(p - 1)^{1/p}}{p \sin(\pi/p)} \pi_p,$$

cf. Lindqvist [11]. Implicitly $S_\rho$ is given by

$$\phi = \int_0^{\pi_p/(p - 1)} \frac{dt}{(1 - t^p/(p - 1))^{1/p}}, \quad \phi \in \left[0, \frac{\pi_p}{2}\right],$$

where $S_\rho(\pi_p/2) = 0$. For $\phi \in (\pi_p/2, \pi_p)$ we define $S_\rho(\phi) = S_\rho(\pi_p - \phi)$ and for $\phi \in (\pi_p, 2\pi_p]$ we set $S_\rho(\phi) = -S_\rho(2\pi_p - \phi)$ and extend $S_\rho$ as a $2\pi_p$-periodic function on $\mathbb{R}$. Generalized sine-functions were discussed in great detail by Lindqvist [11].
Lemma 1. The function $S_p$ satisfies
\[
|u'|^p + |u|^p \frac{p}{p-1} = 1 \quad \text{in } \mathbb{R}, \quad L_p^0 u + u^{(p-1)} = 0 \quad \text{in } \mathbb{R}
\] (8)
and $S_p, |S_p|^p \in C^1(\mathbb{R})$. Furthermore, for $1 < p < 2$ we find $S_p \in C^2(\mathbb{R})$ with $S_p((\mathbb{Z} + 1/2) \pi_p) = 0$, whereas for $p > 2$ the function $S_p$ belongs to $C^2(\mathbb{R}) \setminus (\mathbb{Z} + 1/2) \pi_p$ with $|S_p'|((\mathbb{Z} + 1/2) \pi_p) = \infty$. The function $S_2$ coincides with the standard sine-function, and $\pi_2 = \pi$.

3. THE PRÜFER-TRANSFORMATION.

With the help of the generalized sine-functions we introduce phase-plane coordinates $\rho > 0$ and $\phi$ for a solution $u$ of (6) as follows:
\[
\xi = r^{(p-1)} = \rho S_p'(\phi)^{p-1},
\]
\[
\eta = u^{(p-1)} = \rho S_p(\phi)^{p-1}.
\]

Lemma 2. For a non-trivial solution $u$ of (6) there exists a pair of functions $\rho, \phi$ in $C^1$ that satisfy (9), (10) and the differential equations
\[
\phi' = \frac{r^p}{p-1} \left((q + sy) S_p^* - (q + sv) S_p^- \right) \left(S_p^{(p-1)} - \right)
\]
\[
\rho' = -r^p \left((q + sy) S_p^* - (q + sv) S_p^- \right) |S_p|^p S_p^p
\]
Moreover, $\rho$ and $\phi$ modulo $2\pi_p$ are uniquely determined by $u$. Conversely, a pair of functions $\rho, \phi$ that satisfy (11) and (12) provide a solution $u$ of (6).

Proof. By (8) we find that $\rho^{(p-1)} = |\xi|^{(p-1)} + (1/\rho^{(p-1)}) |\eta|^{(p-1)}$ defines $\rho$ as a positive $C^1$-function as long as $\rho$ does not attain the value 0. We shall see in Theorem 5 that $\rho(r_0) = 0$ implies $u \equiv 0$ for solutions $u$ of (6). Hence for a non-trivial solution $u$ the function $\rho$ is well defined as a $C^1$-function, which never attains the value 0. Likewise, (9) defines $\phi$ as a $C^1$-function modulo $2\pi_p$ as long as $\phi(r) \neq (k + 1/2) \pi_p$, and (10) defines $\phi$ as a $C^1$-function for $\phi(r) \neq k\pi_p$, $k \in \mathbb{N}_0 = \{0, 1, 2, ..., \}$. As a result, every non-trivial solution $u$ of (6) corresponds uniquely to $C^1$-functions $\rho$ and $\phi$ modulo $2\pi_p$. 
Differentiation of (9), (10) gives

\[ r' = r S_{r}^{(r-1)} + (p-1) \rho \ |S_{r}|^{r-2} S_{r}' \phi', \]

\[ (p-1) \ |\rho S_{r}^{(r-1)}|^{(r-2)/(r-1)} e^{-\eta/(r-1)} (\rho S_{r}^{(r-1)})^{1/(r-1)} \]

\[ = \eta' = \rho' S_{r}^{(r-1)} + (p-1) \rho \ |S_{r}|^{r-2} S_{r}' \phi'. \]

Using the identity (8) this reduces to

\[ -r\rho'(q + sx) S_{r}^{+} - (q + sv) S_{r}^{-} \ |S_{r}|^{r-2} = \rho' S_{r}^{(r-1)} - \rho S_{r}^{(r-1)} \phi', \quad (13) \]

\[ (p-1) \rho e^{-\eta/(r-1)} |S_{r}|^{r-2} S_{r}' = \rho' S_{r}^{(r-1)} + (p-1) \rho \ |S_{r}|^{r-2} S_{r}' \phi'. \quad (14) \]

Multiplying (13) with \(-1/(p-1) S_{r}\), (14) with \(1/(p-1) S_{r}^{(r-1)} |S_{r}|^{2-r}\) and adding yields (11). Multiplication of (13) with \(S_{r}'\) of (14) with \(S_{r}/(p-1)\) and adding leads to (12).

Our distinction of regular (R) and singular (S) cases is justified by the equations (11), (12), which indeed become singular at \(r = 0\) when \(a = 0\) and \(\alpha \geq p - 1\). In this case we have the following estimate

**Lemma 3.** In the singular case (S) the argument function \(\phi\) of a solution of (6) on \(I = [0, b]\) satisfies \(\phi(r) - (\pi r/2) = O(r^{\alpha+1})\) as \(r \to 0^+\).

**Proof.** From an integration of (6) we get

\[ |\zeta(t)| \leq \left| \int_{0}^{t} \rho'(q(t) + s(t)v) u^+(t) - (q(t) + s(t)v) u^-(t) \ |u(t)|^{r-2} \ dt \right| \]

\[ \leq A r^{\alpha+1} \quad \text{on } I. \]

Thus, by (9), \(|S_{r}'(\phi(r))|^{r-1} \leq Br^{\alpha+1}\) on \(I\). Using (8) we find

\[ \lim_{\phi \to \pi/2} \left| \frac{S_{r}'(\phi)}{(\pi/2) - \phi} \right|^{(r-1)} = \left| \frac{d}{d\phi} \left( \frac{S_{r}' \left( \frac{\pi}{2} \right)}{(r-1)} \right) \right| = S_{r} \left( \frac{\pi}{2} \right)^{(r-1)} = (p-1)^{(r-1)/r} \]

which implies the claim of the lemma.

The following lemma gives a standard result in the theory of first-order differential inequalities which is widely used in our paper. Since it seems to be not generally known, we outline the proof as given by Walter in [16].

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LEMMA 4 (Comparison Theorem). Assume that \( g(r, s) \) is defined in a set \( \mathcal{D} \subseteq I \times \mathbb{R} \) and satisfies a generalized Lipschitz condition
\[
|g(r, s) - g(r, \tilde{s})| \leq h(r) |s - \tilde{s}|
\]
for \((r, s), (r, \tilde{s}) \in \mathcal{D},\)

where \( h \in L^1(I) \). If the functions \( \phi, \tilde{\phi} \) are absolutely continuous in \( I \) with graph \( \phi, \tilde{\phi} \in \mathcal{D} \) and they satisfy
\[
\phi' \leq g(r, \phi) \quad \text{and} \quad \tilde{\phi}' \geq g(r, \tilde{\phi}) \quad \text{a.e. in } I \quad \text{and} \quad \phi(a) \leq \tilde{\phi}(a),
\]
then \( \phi \leq \tilde{\phi} \) in \( I \), more precisely,

(i) \( \phi < \tilde{\phi} \) in \((a, b]\), or

(ii) there is \( c \in (a, b] \) such that \( \phi \equiv \tilde{\phi} \) in \([a, c] \) and \( \phi < \tilde{\phi} \) in \((c, b] \).

The functions \( \phi \) and \( \tilde{\phi} \) are called sub- and supersolutions, resp.

Proof. The difference \( \psi = \tilde{\phi} - \phi \) satisfies \( \psi(a) \geq 0 \) and
\[
\psi' \geq g(r, \tilde{\phi}) - g(r, \phi) \geq -h(r) |\psi| \quad \text{in } I.
\]

This shows that \( \psi e^{-\int_{L^0}^L |g(r, s)| ds} \) is increasing on intervals where \( \psi \) is negative. Since \( \psi(a) \geq 0 \) we obtain that \( \psi \) is non-negative in \( I \). Likewise, \( \psi e^{\int_{L^0}^L |g(r, s)| ds} \) is increasing on intervals where \( \psi \) is non-negative. This implies that either \( \psi > 0 \) on \((a, b]\) or, if \([a, c]\) is the largest interval where \( \psi \) vanishes identically, that \( \psi > 0 \) on \((c, b]\). This proves the lemma.

THEOREM 4. Let \( u, \tilde{u} \) be solutions of
\[
L_p^s u + q u^{(p-1)} + s(\mu(u^+)^{p-1} - v(u^-)^{p-1}) = 0 \quad \text{in } I,
\]
\[
L_p^s \tilde{u} + \tilde{q} \tilde{u}^{(p-1)} + \tilde{s}(\tilde{\mu}(\tilde{u}^+)^{p-1} - \tilde{v}(\tilde{u}^-)^{p-1}) = 0 \quad \text{in } I,
\]
with \( q(r) \leq \tilde{q}(r), s(r) \leq \tilde{s}(r) \) in \( I \) and \( \mu \leq \tilde{\mu}, v \leq \tilde{v} \). Then the argument functions \( \phi, \tilde{\phi} \) satisfy
\[
\phi(a) \leq \tilde{\phi}(a) \Rightarrow \phi < \tilde{\phi} \quad \text{in } (a, b] \quad \text{or} \quad \phi \equiv \tilde{\phi} \quad \text{in } [a, c], \quad \phi < \tilde{\phi} \quad \text{in } (c, b].
\]

Proof. Notice that the functions \( S_p^s S_r^{(p-1)} - S_p^s S_r^{(p-1)} + |S_p^s|^p, \) which appear in (11), are non-negative \( C \)-functions. Therefore, since \( \phi \) satisfies (11) and \( \tilde{\phi} \) satisfies the corresponding equation with \( q, s, \mu, v \) replaced by \( \tilde{q}, \tilde{s}, \tilde{\mu}, \tilde{v} \), we find that \( \phi \) is a supersolution to (11).

In the regular case (R) the theorem follows from Lemma 4 since the right-hand side of (11) satisfies a generalized Lipschitz-condition. In the singular case (S) we know from Lemma 3 that \( |\phi(r) - (\pi_p/2)| \leq Ar^{s+1} \) for \( r \in I \). By (15) \( |S_p^s(\phi)|^p \leq K |\phi - (\pi_p/2)|^{p(\alpha-1)} \), and we can choose \( A, K \) such that the same estimate holds for \( \tilde{\phi} \). If we define \( g(r, \phi) = r^{-s(p-1)} |S_p^s(\phi)|^p, \) then we find for \( |\phi - (\pi_p/2)| \leq Ar^{s+1} \) that
\[ |\partial_\varphi g(r, \varphi)| = p |S'_p(\varphi)|^p |S''_p(\varphi)| r^{-\pi(p-1)} \]
\[ \leq K_1 \left| \varphi - \frac{\pi_p}{2} \right|^{1/(p-1)} r^{-\pi(p-1)} \]
\[ \leq K_2 r^{1/(p-1)} \quad \text{for} \quad r \in I. \]

In the domain \( \mathcal{D} = \{(r, \varphi) : |\varphi - (\pi_p/2)| \leq A r^{x+1}, 0 \leq r \leq b\} \) in the \((r, \varphi)\)-plane where the graphs of \( \phi, \phi' \) take their values, the function \( g(r, \varphi) \) is Lipschitz-continuous w.r.t. \( \varphi. \) Hence the assertion follows again from Lemma 4.

In the context of the 1-dimensional \( p \)-Laplacian \((x = 0)\) phase-space transformations using generalized sine-functions appeared in the work of Naito [12] and Fabry, Fayyad [7]. In order to make full use of the Prüfer-transformation it is important to get the right form of the transformation (9), (10) which makes the equation (11) for \( \phi \) independent on \( \rho \) even in the higher-dimensional cases \( x > 0. \)

4. A-PRIORI BOUNDS AND THE INITIAL VALUE PROBLEM

**Lemma 5.** Let \( I = [a, b] \) and \( I^0 = (a, b). \) If \( F(r, t) \) is increasing in \( t \) and
\[ |L^*_p v| \leq F(r, |v|) \quad \text{in} \quad I^0, \]
\[ L^{*}_p w \geq F(r, w) \quad \text{in} \quad I^0 \]
and \( |v(a +)| < w(a +), |v'(a)| \leq w'(a) \) then \( |v'| \leq w' \) and \( |v| < w \) in \((a, b]\).

**Proof.** Let \( |v| \leq w \) in \( I' = [a, c] \) with \( c > a \) maximal. Let \( V = r^x v^{(p-1)}, \)
\[ W = r^x w^{(p-1)}. \]
Since \( V' \leq W' \) in \( I' \) and \( V(a) \leq W(a) \) we have \( V \leq W \) in \( I' \)
and hence \( v' \leq w' \) in \( I' \). The same reasoning applies to \(-v\) instead of \( v \) since \( L^*_p (-v) = -L^*_p v. \) It shows that \(-v' \leq w' \) in \( I' \) and \(-v < w \) in \((a, c]\). Thus \( c = b \) and the result is proved.

**Corollary 1.** Suppose there are a function \( u \) and positive constants \( A, B, C \) such that
\[ |L^*_p u| \leq A |u|^{p-1} + B \quad \text{in} \quad I, \]
\[ |u(a)|, |u'(a)| \leq C. \]
Then \(|u(r)| \leq Ce^{Er}, \ |u'(r)| \leq CEe^{Er}\) in \(I\), where \(E \geq 1\) is so large that 
\[E(p - 1) \geq A + B/C^{p-1}.\]

**Proof.** Let \(F(r, t) = At^{p-1} + B\) and \(w = Ce^{Er}\). Then 
\[L_p^w w = (CE)^{p-2} e^{(p-2)Er} \left((p-1) CE^2 e^{Er} + \frac{2}{r} CE e^{Er}\right)\]
\[\geq (p-1) C^{p-1} Er^{p-1} e^{Er}\]
\[\geq (AC^{p-1} + B) e^{(p-1)Er}\]
\[\geq Aw^{p-1} + B.\]

By Lemma 5, we obtain the desired estimate. \(\blacksquare\)

**Remark.** Corollary 1 and the uniqueness conditions on \(f\) imply the existence in the entire interval \(I\) and the uniqueness of solutions of the initial value problems for (1) as described in Theorem 3.

For the standard homogeneous eigenvalue equation (4) uniqueness for the initial value problem was proved by Walter in [15]. Here we present a uniqueness result for the more general homogeneous differential equation (6).

**Theorem 5.** For \(a \geq 0\) and \(c, d \in C([a, b])\) the initial value problem 
\[L_p^a u + (c(r) u^+ - d(r) u^-) |u|^{p-2} = 0 \quad \text{in} \ [a, b],\]
\[u(0) = u_0, \quad u'(0) = 0 \quad \text{in case (S)}\]  \[\quad (r^a u'(p-1))(a) = u_0' \quad \text{in case (R)}\]
has a unique solution in \([a, b]\).

**Proof.** A solution \(u\) of the initial value problem satisfies 
\[r^a u'(r)^{p-1} = u'_0 - \int_a^r r^a (c(t) u^+)^{p-1} - (r^a d(t) u^-)^{p-1} \, dt\]  \[\tag{17}\]
and is obtained as a continuous solution of the fixed point equation 
\[u(r) = u_0 + \int_a^r \left( u'_0 \right)^{r^a \int_a^r c(t) (u^+)^{p-1} - (r^a d(t) (u^-)^{p-1}) \, dt\right)^{(1/p-1)} \, dr.\]

Local existence follows from Schauder’s fixed point theorem as in Walter [15], and global existence follows from Corollary 1. Next we prove uniqueness. The case \(u_0 = u_0' = 0\) is the simplest. In this case we get from the integral equation the estimate 
\[|u(r)| \leq K \max_{a \leq r \leq r_0} |u| \int_a^r (r^a)^{(1/p-1)} \, dt \quad \text{for} \quad a \leq r \leq r_0.\]
where $K$ is a positive constant. Hence $u \equiv 0$ in a sufficiently small interval $[a, a + \varepsilon]$. Iterating this argument implies $u \equiv 0$ in $[a, b]$. Thus, a non-trivial solution of the initial value problem has a discrete set of zeroes, which are simple. Therefore it suffices to show local uniqueness at $r_0 \geq a$ for the cases shown in the following table. As before, (R) refers to the regular case $r_0 > 0$ or $r_0 = 0$ and $0 \leq x < p - 1$, and (S) refers to the singular case $r_0 = 0$ and $x \geq p - 1$. With $U_h$ we denote the local uniqueness of the (homogeneous) initial-value problem (16) and with $U_i$ we denote the local uniqueness of the inhomogeneous initial-value problem, where a continuous function $h = h(r)$ is on the right-hand side of (16).

<table>
<thead>
<tr>
<th>Initial values</th>
<th>$1 &lt; p \leq 2$</th>
<th>$p &gt; 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(R)</td>
<td>(S)</td>
</tr>
<tr>
<td>(i)</td>
<td>$u_0 \neq 0$, $u'_0 \neq 0$</td>
<td>$U_i$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$u_0 = 1$, $u'_0 = 0$</td>
<td>$U_i$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$u_0 = 0$, $u'_0 = 1$</td>
<td>$U_h$</td>
</tr>
</tbody>
</table>

With $v = r^\alpha u^{\frac{1}{(p-1)}}$ the inhomogeneous version of (16) can be written as the system

\begin{align*}
u' &= r^{-x(p-1)\frac{1}{(p-1)}}u, \\
v' &= r^\alpha h(r) - r^\alpha(c(u^+)^{p-1} - d(u^-)^{p-1}).
\end{align*}

For $1 < p \leq 2$ the right-hand side of the first equation is Lipschitz-continuous with respect to $v$, and for $p > 2$ the second equation is Lipschitz-continuous with respect to $u$. Thus, the cases (i) for all $p$, (ii)(R) for $1 < p \leq 2$ and (iii)(R) for $p > 2$ follow by the classical Lipschitz-Caratheodory uniqueness. Case (ii)(S) for $1 < p \leq 2$ follows from Theorem 4 case (ii)(iii) in [13]. The homogeneous case (ii)(S) for $p > 2$ is proved in Walter [15], and precisely the same proof holds for (ii)(R) and $p > 2$. The counterexample to uniqueness in the inhomogeneous case is also found in Walter [15]. To prove uniqueness in case (iii)(R) for $1 < p \leq 2$, we observe that the Prüfer-transformed equation (11) for the argument-function $\phi$ has a right-hand side that is Lipschitz-continuous in $\phi$. Hence up to translations modulo $2\pi_p$ there is a unique solution of (11) with $\phi(r_0) = 0$ modulo $2\pi_p$. The uniqueness of $\phi$ implies by (12) the uniqueness of $p$ with $p(r_0) = 1$. Therefore uniqueness of $u$ follows.
Remark. Uniqueness in case (iii) (R) for $1 < p \leq 2$ also holds in the inhomogeneous case. We presented the proof for the homogeneous case only, because we wanted to show the usefulness of Prüfer-transform in the context of uniqueness questions. In fact, the only tool known to us that allows to prove uniqueness in the homogeneous case (ii) for $p > 2$ is the Prüfer-transformation. The usual method of investigating the difference of two solutions fails, because this method does not distinguish between the homogeneous and the inhomogeneous case where uniqueness may fail.

Lemma 6. Let $\phi(r; \mu, v)$ be the argument function of a solution of (6) with initial-values $\phi(a; \mu, v) = \gamma$, where $\gamma \in [0, \pi_1)$ is such that

$$\gamma = \frac{\pi}{2} \quad \text{in case (S),}$$

$$\gamma_1 S_\mu(\gamma)^{(p-1)} + \gamma_2 S'_\mu(\gamma)^{(p-1)} = 0 \quad \text{in case (R).}$$

Then $\phi(r; \mu, v)$ is a $C^1$-function in $\mu$ and $v$.

Proof. The differential equation (11) for $\phi$ takes the form $\phi' = r^p g_1(\phi, \mu, v) + r^{-n(p-1)} g_2(\phi)$, where $\partial_\phi g_1$, $\partial_\mu g_1$, $\partial_v g_1$ and $g_2$ are continuous. In the regular case (R) the standard differentiability theorem, cf. Walter [16], gives the result. In the singular case (S) we consider $(\mu, v)$ in a bounded open set $K \subset \mathbb{R}^2$. By Lemma 3 there exists a constant $A = A(K) > 0$ such that $|\phi(r; \mu, v) - (\pi_\mu/2)| \leq Ar^{n^*+1}$ for all $r \in [a, b]$ and $(\mu, v) \in K$. For $|t - (\pi_\mu/2)| \leq Ar^{n^*+1}$ we find that $r^{-n(p-1)} g_2'(t) = r^{-n(p-1)} |S'_\mu(t)|$ extends continuously to $(r, t) = (a, 0)$ since $(|S'_\mu|^{p'})$ is of order $r^{(p+1)(p-1)}$ by Lemma 1, Lemma 3 and (15). Hence, if we define $h_2(r, t) = r^{-n(p-1)} g_2(t)$ for $(r, t) \in [a, b] \times \mathbb{R}$ with $|t - (\pi_\mu/2)| \leq Ar^{n^*+1}$ and if we extend $h_2(r, t)$ for $t \geq (\pi_\mu/2) + Ar^{n^*+1}$ and $t \leq (\pi_\mu/2) - Ar^{n^*+1}$ such that $h_2$ is continuous in $[a, b] \times \mathbb{R}$ and for fixed $r \in [a, b]$ the values $\partial_t h_2$ are constant in $t$ for $t \geq (\pi_\mu/2) + Ar^{n^*+1}$ and $t \leq (\pi_\mu/2) - Ar^{n^*+1}$, then $h_2$ is a $C^1$-function in $[a, b] \times \mathbb{R}$. Moreover, for $(\mu, v) \in K$ the function $\phi = \phi(\cdot; \mu, v)$ satisfies $\phi' = r^p g_1(\phi, \mu, v) + h_2(r, \phi)$. Therefore differentiability with respect to $\mu, v$ follows from the standard theory.

5. EXISTENCE OF EIGENVALUES.

Let $\phi(r, z)$ be the argument function for an initially positive solution $u$ of the eigenvalue equation (4) which satisfies $(Bu)(a) = 0$. Then $\phi$ solves

$$\phi' = \frac{r^p}{p-1} |S_\mu(\phi)|^p + r^{-n(p-1)} |S'_\mu(\phi)|^p \quad (21)$$
with \( \phi(a, \lambda) = \gamma \), where \( \gamma \in [0, \pi_p) \) is determined by (20). By Theorem 4 the function \( \phi(\cdot; \lambda) \) is strictly increasing in \( \lambda \). Moreover we have

**Lemma 7.** (i) \( \phi(b; \lambda) \to 0 \) for \( \lambda \to -\infty \). (ii) \( \phi(b; \lambda) \to \infty \) for \( \lambda \to \infty \).

**Proof.** Notice that \( \phi(r; \lambda) > 0 \) on \([a, b]\) because \( \phi' > 0 \) at points where \( \phi = k\pi_p, \, k \in \mathbb{Z} \). Hence \( \phi \) crosses the lines \( k\pi_p \) only from below (and hence only once).

(i) **Case** \( a > 0 \). We choose a linear function \( \psi \) such that \( \psi(a) = \pi_p - \varepsilon > \gamma \) and \( \psi(b) = \varepsilon \), where \( \varepsilon > 0 \) is sufficiently small. Since \( S_p(\psi) \geq \delta > 0 \) on \( I \) it is easy to see, that \( \psi \) is an upper solution to (21) if \( \lambda \) is sufficiently negative. By the standard comparison argument of Lemma 4 applied to \( \phi, \psi \) we get \( 0 < \phi(b; \lambda) \leq \psi(b) = \varepsilon \). This proves (i).

Case \( a = 0 \). Let \( \lambda \) be so small that \( \eta + s \lambda \leq 0 \) in \( I \). If \( \pi \geq p - 1 \) we get from (21) and Lemma 3 the estimate \( \phi' \geq r^{-\pi(p-1)} \left| S_p'(\phi) \right|^p \leq K r^{\pi + p(p-1)} \) in \( I \). If \( 0 \leq \pi < p - 1 \) we obtain \( \phi' \leq r^{-\pi(p-1)} \), which is integrable. For sufficiently small \( \varepsilon > 0 \) we find in both cases a value \( \eta > 0 \) such that

\[
\phi(q; \lambda) \leq \phi(0; \lambda) + \varepsilon = \gamma + \varepsilon \leq \pi_p - \varepsilon.
\]

Using a linear function \( \phi \) with \( \phi(q) = \pi_p - \varepsilon, \, \phi(b) = \varepsilon \) we find \( 0 < \phi(b; \lambda) \leq \varepsilon \) as before.

(ii) For comparison purposes we consider the following initial value problem with constant coefficients

\[
\tau_0 L_0 u_0 + 2 \sigma_0 u_0^{(p-1)} = 0 \quad \text{on} \quad [a + \eta, b],
\]

\[
u_0(a + \eta) = 0, \quad u_0(a + \eta) > 0,
\]

where \( \tau_0, \sigma_0 \) will be chosen later. Up to multiples the solutions are known to be \( u_0(r) = \sin((\lambda \sigma_0/\tau_0)^{1/p} (r - a - \eta)) \). In analogy to (9), (10) we define Prüfer-variables for \( u_0 \) by

\[
\tau_0 u_0'^{(p-1)} = \rho_0 S_p' \phi_0^p, \quad u_0'^{(p-1)} = \rho_0 S_p' \phi_0^p.
\]

By a calculation similar to the one in Lemma 2, we find

\[
\phi_0' = \sigma_0^p \frac{\lambda}{p - 1} |S_p|^p + |S_p'|^p \tau_0^{-(p-1)}. \]

As argument function we may choose \( \phi_0 = (\lambda \sigma_0 / \tau_0)^{1/p} (r - a - \eta) \). For \( \lambda \) sufficiently large, we have \( (\lambda/2) \text{ess inf}_s s \leq q + \lambda s \) on \( I \). Thus, choosing \( \sigma_0 = ((a + \eta)^p/2) \text{ess inf}_s, \tau_0 = b^+ \) we find that \( \phi_0 \) is a subsolution to (21)
on \([a + \eta, b]\). Since \(\phi(a + \eta) > 0\) the standard comparison argument, cf. Lemma 4, yields \(\phi(r; \lambda) > \phi_d(r) = (\lambda \sigma_\eta/\sigma_0)^{1/p} (r - a - \eta)\) on \([a + \eta, b]\). In particular \(\phi(b; \lambda) \to \infty\) as \(\lambda \to \infty\).

**Proof of Theorem 1.** The eigenvalues \(\lambda_n\) are obtained from the equation

\[
\phi(b; \lambda_n) = \delta + (n - 1) \pi_p, \quad n = 1, 2, ..., (22)
\]

where \(\delta \in (0, \pi_p]\) is such that

\[
\gamma_3 S_p(\delta)^{(p-1)} + \gamma_4 S_p(\delta)^{(p-1)} = 0. (23)
\]

Clearly, the \(n\)-th eigenfunction has \((n - 1)\)-zeroes in \(I_0\) since \(\phi\) crosses each line \(kr\) exactly once from below for \(k = 1, ..., n - 1\). The separation properties for consecutive eigenfunctions follows easily from the strict monotonicity of \(\phi(r; \lambda)\) w.r.t. \(\lambda\) and Theorem 4.

**6. THE FUČÍK-SPECTRUM.**

In this section we prove Theorem 2. We begin with a lemma that provides lower bounds for the Fučík-spectrum.

**Lemma 8.** If \((\mu, v) \in \sigma^+\) then \(\mu \geq \lambda_1\). Likewise, if \((\mu, v) \in \sigma^-\) then \(v \geq \lambda_1\).

**Proof.** Suppose \(v\) is an initially positive Fučík-eigenfunction corresponding to \((\mu, v) \in \sigma^+\) with argument function \(\phi\). We assume for contradiction that \(\mu < \lambda_1\). If \(v\) is a first eigenfunction corresponding to the first eigenvalue \(\lambda_1\) with argument function \(\psi\) and \(\phi(a) = \psi(a) \in [0, \pi_p]\), then the Comparison Theorem 4 implies that \(\phi \leq \psi \in [0, \pi_p]\). Since \(\phi\) is also positive in \(I_0\), we find that \(u\) is positive in \(I_0\). This implies that \(u\) is a first eigenfunction and \(\mu = \lambda_1\) in contradiction to our assumption.

**Proof of Theorem 2.** The Fučík-spectrum is clearly closed. From Theorem 5 we know that a Fučík-eigenfunction is either initially positive or initially negative and it is easy to verify that \((\mu, v, u)\) solves the eigenvalue problem (6), (7) if and only if \((v, \mu, -u)\) solves (6), (7). It is clear from Theorem 1 that \((\mu, v) \in \{(\lambda, \lambda) : \lambda \in \mathbb{R}\}\) if and only if \(\mu = v = \lambda_k\) for some \(k \in \mathbb{N}\). For the rest of the proof we only consider \(\sigma^+\). Let \(\bar{u}\) be an initially positive Fučík-eigenfunction with \((k - 1)\)-zeroes in \(I_0\) \((k \geq 2)\). With \((\bar{\mu}, \bar{v})\) we denote the corresponding Fučík-eigenvalue and with \(\bar{\phi}\) the argument function of \(\bar{u}\). To find Fučík-eigenvalues near by, we need so solve

\[
\phi' = \frac{r^p}{p-1} \left( (q + s\lambda) S_p^+ - (q + s\upsilon) S_p^- \right) S_p^{(p-1)} + r^{-\eta(p-1)} |S_p'|^p, \quad \phi(a) = \gamma (24)
\]
and determine $(\mu, \nu)$ such that $\phi(b) = \delta + (k - 1)\pi_p$. The solution $\phi(r) = \phi(r; \mu, \nu)$ of (24) is continuously differentiable w.r.t $\mu, \nu$ by Lemma 6, and $\phi_v = \partial \phi/\partial \nu$ solves

$$
\phi''_v = \frac{p}{p-1} r^\gamma (q + s\nu)(S_p^+)^{p-1} - (q + sv)(S_p^-)^{p-1}) S_p^\nu \phi_v 
+ r^{-\alpha/p-1}(S_p^+)^\nu \phi_v + \frac{r^\alpha}{p-1} s(S_p^-)^\nu
$$

(25)

with $\phi(a) = 0$. In case (R) where either $a > 0$ or $a = 0$ and $0 < a < p - 1$ this initial value problem is regular. In the case (S) where $a = 0$ and $a > p - 1$, the coefficient $(S_p^+)^\nu$ of order $r^{(a+1)/(p-1)}$ by Lemma 1, Lemma 3 and (15). Thus, also in this case, the initial value problem for $\phi_v$ is regular. The solution $\phi_v$ is seen to be non-negative in $[a, b]$. Since $\phi$ corresponds to a solution with $1 \leq (k - 1)$-zeroes in $I^p$, it crosses the lines $in_p$ for $i = 1, \ldots, k - 1$, and hence $\int_I (r^\alpha/(p-1)) s(S_p^-)^\nu dr > 0$. This implies in particular $\phi_v(b) > 0$ (and by a similar argument $\phi_v(b) > 0$). Hence the implicit function theorem applies and provides locally around $\mu$ a unique $C^1$-solution curve $v = v(\mu)$ such that

$$
\phi(b; \mu, v(\mu)) = \delta + (k - 1)\pi_p, \quad v(\mu) = \bar{v}.
$$

We extend this local $C^1$-curve by the implicit function theorem to its maximal interval of definition $I_k$, which we assume to include $\lambda_k$. We observe that $v(\mu)$ is strictly decreasing in $\mu$, and in fact $v' = - (\phi_v)^{-1} \phi_v < 0$.

Next we show that $v(\mu) > \lambda_1$ for $\mu$ in $I_k$. By the bounds on the Fučík-spectrum from Lemma 8 we know that $I_k$ has a finite left endpoint $\mu_{\pi_p} \geq \lambda_1$ with $\lim_{\mu \to \mu_{\pi_p}} v(\mu) = \infty$. Thus, if we suppose for contradiction that $v(\mu) \leq \lambda_1$ at some point in $I_k$, then there exists $\bar{\mu} \in I_k$ with $v(\bar{\mu}) = \lambda_1$. Let $\psi$ be the argument function of the first eigenfunction such that $\psi(0) = \gamma$ and let $r_1 \in I^p$ be the first zero of $\phi_{\mu, \pi_p}$. Notice that $\psi + \pi_p$ is also an argument function for the first eigenfunction and $\psi(r_1) + \pi_p \in (\pi_p, 2\pi_p)$. In particular $\psi(r_1) + \pi_p > \phi_{\mu, \pi_p}(r_1) = \pi_p$. By the standard comparison argument we find $\psi + \pi_p > \phi_{\mu, \pi_p}$ on $(r_1, b]$. Therefore $\phi_{\mu, \pi_p}$ does not satisfy the boundary condition at $b$, and we have reached a contradiction to the assumption $v(\mu) \leq \lambda_1$ somewhere in $I_k$.

Now it follows from the monotonicity of $v(\mu)$ and the bounds on the Fučík-spectrum that the maximal domain of definition for $v(\mu)$ is a halfline $(\mu_{\pi_p}, \infty)$ with $\lim_{\mu \to \mu_{\pi_p}} v(\mu) = \infty$ and $\lim_{\mu \to \infty} v(\mu) = v_{\infty}$, where $\mu_{\pi_p}, v_{\infty}$ will be determined in the sequel. Before we do that, suppose that there are

1 Here $\gamma, \delta$ are determined by (20), (23) as before.
two curves \( v_1, v_2 \) such that \( (\mu, v_1(\mu)), (\mu, v_2(\mu)) \in \sigma^+_k \). Since \( v_1(\lambda_k) = v_2(\lambda_k) = \lambda_k \), the uniqueness part of the implicit function theorem implies \( v_1 \equiv v_2 \).

Hence the component \( \sigma^+_k \) consists of exactly one curve \( v \). The asymptotics of \( v \) will easily follow from the next lemma.

**Lemma 9.** Let \( u_{\mu, v(\mu)} \) be the Fučik-eigenfunction corresponding to \( (\mu, v) \in \sigma^+_k \) for \( k = 2i \pm 1 \) or \( k = 2i (i \in \mathbb{N}) \). Let \( \phi = \phi_{\mu, v(\mu)} \) be its argument function with \( \phi(a) = \gamma \in [0, \pi_k), \phi(b) = \delta + (k - 1) \pi_p \) (\( \delta \in (0, \pi_p) \)). Let \( r_j \in (a, b) \) be the \( j \)-th interior zero of \( u \), i.e., \( \phi(r_j) = \pi_{2j} \). In the singular case \( (S) \), we also need to define \( r_0 \) as the first point where \( \phi = 3\pi_p/2 \).

Then there exists \( \varepsilon = \varepsilon(k) > 0 \) such that the following holds:

For \( \mu \to \infty \):

\[
\begin{align*}
& r_1 - a \to 0 \quad \text{in case (R)}, \\
& b - r_{k-1} \to 0 \quad \text{if } k \text{ is odd}, \\
& b - r_{k-1} \geq \varepsilon \quad \text{if } k \text{ is even},
\end{align*}
\]

and the distance of any odd-numbered zero to its preceding even-numbered zero tends to zero. The distance of any even-numbered zero to its preceding odd-numbered zero is \( \geq \varepsilon \).

For \( v \to \infty \):

\[
\begin{align*}
& r_1 - a \geq \varepsilon, \\
& b - r_{k-1} \to 0 \quad \text{if } k \text{ is even}, \\
& b - r_{k-1} \geq \varepsilon \quad \text{if } k \text{ is odd},
\end{align*}
\]

and the distance of any even-numbered zero to its preceding odd-numbered zero tends to zero. The distance of any odd-numbered zero to its preceding even-numbered zero is \( \geq \varepsilon \).

**Proof.** The limit \( \mu \to \infty \): Let \( \mu \) be so large, that \( q + s\mu \geq \mu/2 \) in \( I \).

First we consider the case \( a = 0 \) and \( 0 \leq \alpha < p - 1 \). Let \( \tilde{u} \) be the solution of the initial value problem \( L^p_{\alpha} u + u^{(p-1)} = 0, \quad u(0) = 0, \quad (r^{\alpha(p-1)}u')(0) = 1 \) and let \( \tilde{u}(r) = \tilde{u}(2/\mu)^{1/p} r \). Then \( \tilde{u} \) satisfies

\[
L^p_{\alpha} \tilde{u} + \frac{2}{r^{p-1}} u^{(p-1)} = 0, \quad \tilde{u}(0) = 0, \quad (r^{\alpha(p-1)}u')(0) > 0. \tag{26}
\]

The corresponding argument function \( \tilde{\phi} \) with \( \tilde{\phi}(0) = 0 \) is a subsolution to \( \phi \) on \( [0, r_1] \). Hence \( r_1 \to 0 \) as \( \mu \to \infty \). On \( [r_1, r_2] \) the function \( \phi \) satisfies

\[
\phi' = \frac{r^p}{p-1} (q + s\mu) |S_p|^p + |S_p'|^p r^{-\alpha(p-1)}, \tag{27}
\]
where $v(u) \sim v_\infty \geq \lambda_1$ as $\mu \to \infty$. Thus it is easy to see that there exists $\varepsilon > 0$
with $r_2 - r_1 \geq \varepsilon$. Likewise, on $[r_{2j}, r_{2j+1}]$ the function $\psi(r) = \beta(r - r_{2j}) + 2j\pi_p$ with $\beta = \max\{|r_{2j+1}^{2}/2, b^{-n(p-1)}\}$ is a subsolution to $\phi$ and thus $r_{2j+1} - r_{2j} \to 0$ as $\mu \to \infty$ and (with an argument similar to $r_2 - r_1 \geq \varepsilon$) we also find $r_{2j} - r_{2j-1} \geq \varepsilon$. If $k$ is odd we use $\psi(r) = \beta(r - r_{k-1}) + (k - 1)\pi_p$ with $\beta = \max\{|r_{k-1}^{2}/2, b^{-n(p-1)}\}$ as a subsolution on $[r_{k-1}, b]$ to see that $b - r_{k-1} \to 0$ as $\mu \to \infty$. If $k$ is even, then $S_k(\phi)$ is negative in $(r_{k-1}, b)$ and (27) holds, which shows that $b - r_{k-1} \geq \varepsilon$ for some $\varepsilon > 0$.

Next we consider the case $a > 0$. Apart from $r_1 \to a$ the argument is as before. To see the latter, observe that $\phi(r) = \beta(r - a) + \gamma$ is a subsolution to $\phi$ on $[a, r_1]$ if $\beta = \max\{|a^{2}/2, b^{-n(p-1)}\}$. Hence the conclusion $r_1 \to a$ as $\mu \to \infty$.

In the singular case (S) where $a = 0$ and $x \geq p - 1$ the arguments are again the same apart from the fact that this time $\tilde{\tau}_1 \to 0$ as $\mu \to \infty$. To verify this notice first that $r_1 \to 0$. The proof is as in the regular case $0 \leq \alpha < p - 1$ where $\tilde{a}$ now has the initial values $\tilde{u}(0) = 1$, $\tilde{u}'(0) = 0$. Next we define $\psi$ as the solution of

$$
\psi' = \frac{1}{\pi} |S_\mu(\psi)|^{\rho} r^{-n(p-1)}, \quad \psi(r_1) = \pi_p.
$$

The function $\psi$ is defined on $[r_1, \infty)$ and it is given implicitly by

$$
\int_{r_1}^{\psi(r)} \frac{2 \, dt}{|S_\mu(t)|^{\rho}} = \begin{cases} \left( 1 - \frac{x}{p-1} \right) (r_1^{1-n(p-1)} - r_{1}^{-n(p-1)}), & x > p-1, \\ \log r - \log r_1, & x = p-1. \end{cases}
$$

Clearly the value $\psi(2r_1)$ tends to $3\pi_p/2$ as $r_1 \to 0$. Since $|S_\mu(t)|^{\rho}|t - 3\pi_p/2|^{p(p-1)}$ is equal to $t \to 3\pi_p/2$ (cf. (15)), we find constants $K_1, K_2 > 0$ with $K_1 |t - 3\pi_p/2|^{p(p-1)} = \varepsilon \leq |S_\mu(t)|^{\rho} \leq K_2 |t - 3\pi_p/2|^{p(p-1)}$ for $t \in [\pi_p, 2\pi_p]$. Therefore we can determine a constant $K > 0$ such that

$$
K \int_{r_1}^{\psi(r)} |t - 3\pi_p/2|^{-p(p-1)} \, dt \leq \begin{cases} \left( r_1^{1-n(p-1)} - r_1^{-n(p-1)} \right), & x > p-1, \\ \log r - \log r_1, & x = p-1. \end{cases}
$$

Evaluating the integral and defining $\kappa = K(p-1)(\pi_p/2)^{1/(1-p)}$ gives

$$
\frac{3\pi_p}{2} - \psi(r) \geq \begin{cases} L(r_1^{1-n(p-1)} - r_1^{-n(p-1)} + \kappa)^{1-p}, & x > p-1, \\ L(\log(r/r_1) + \kappa)^{1-p}, & x = p-1, \end{cases}
$$

for some $L > 0$. This implies the following estimate for $r \in (r_1, 2r_1)$ if $r_1$ is sufficiently close to $0$

$$
|S_\mu(\psi(r_1))|^{\rho} \geq \begin{cases} L_1(r_1^{1-n(p-1)} - r_1^{-n(p-1)} + \kappa)^{-p}, & x > p-1, \\ L_2 r^{n(p-1)} - p, & x = p-1. \end{cases}
$$

(28)
for some $L_1, L_2 > 0$. If we choose $\mu$ large enough, then $r_1$ becomes so small that the following estimate holds for $r \in [r_1, 2r_1]$

$$r^\beta (\|q\|_\infty + \|s\|_\infty |\lambda_1|) \lesssim r^{-\beta (p-1)} \frac{|S'_q(\psi(r))|^p}{2}.$$  

Then it is easy to verify that $\psi$ satisfies

$$|\psi| \lesssim r^{\beta} (q + sv_\mu) |S_p(\psi)|^r + |S'_p(\psi)|^p r^{-\beta (p-1)} \quad \text{on} \quad [r_1, 2r_1].$$

This means that $\psi$ is a subsolution to $\phi$ on $[r_1, 2r_1] \cap [r_1, r_2]$. Hence, the point $r_1$, where $\phi$ first reaches $\psi(2r_1)$, lies in $[r_1, 2r_1]$ and tends therefore to 0 as $\mu \to \infty$. Since $\psi(2r_1)$ tends to $3\pi \mu / 2$, we find that $r_1 \to 0$ as $\mu \to \infty$.

**The limit $v \to \infty$.** The statements in this case are verified in a similar way as before with the help of comparison argument functions. Since the Fučik-eigenfunctions are initially positive, it is easy to verify that $r_1 - a \geq \epsilon$ for some positive $\epsilon$. Thus, the possible singularity at $a=0$ does not play any role in the construction of comparison functions $\psi$ like in the case $a>0$ of the previous proof for $\mu \to \infty$.

Now we can finish the proof of Theorem 2.

**The limit $\mu \to \infty$.** In the regular case (R) the previous lemma shows the following about the interior zeroes of $u_{\mu, \nu(\mu)}$; the first zero tends to $a$, the odd numbered zeroes and the preceeding even numbered zeroes join, the last zero tends to $b$ if $k$ is odd and stays away from $b$ if $k$ is even. Hence, we may extract a subsequence $\mu_i \to \infty$ such that $\mu_i \to \infty$ such that $r_1 \to a = R_1, r_2, r_3 \to R_2, r_4, r_5 \to R_3, ..., r_{k-2} \to R_{k-1} < b = R_{k+1} (k = 2i), \text{and} r_{k-3}, r_{k-2} \to R_{k}, r_{k-1} \to b = R_{k+1} (k = 2i+1)$ and $\phi(\cdot; \mu_i, v(\mu_i)) \to \phi_{\infty}(\cdot)$ as $l \to \infty$ locally uniformly on the nodal intervals $(R_1, R_2), (R_2, R_3), ..., (R_{i}, R_{i+1})$ where $\phi_{\infty}$ satisfies for $j = 2, ..., i+1$

$$\phi_{\infty} = r^{\beta} (q + sv_\infty) |S_p|^r + r^{-\beta (p-1)} |S'_p|^p \quad \text{on} \quad (R_{j-1}, R_j). \quad (29)$$

Moreover we find that $\phi_{\infty}(R_j \pm) = (2j-1)\pi_p$ for $j = 1, ..., i$, $\phi_{\infty}(R_j \pm) = (2j-2)\pi_p$, $j = 2, ..., i$ and $\phi(R_{i+1} \pm) = 2i\pi_p$ if $k$ is odd and $\phi(R_{i+1} \pm) = (2i-1)\pi_p + \delta$ if $k$ is even. If we define

$$\tilde{\phi}_{\infty} = \phi_{\infty} - (j-1)\pi_p \chi(R_{j-1}, R_j), \quad j = 2, ..., i+1, \quad (30)$$

then $\tilde{\phi}_{\infty}$ is a $C^1$-solution of (29) with $\tilde{\phi}_{\infty}(a) = 0, \tilde{\phi}_{\infty}(b) = (i-1)\pi_p + \delta$ if $k$ is even, and $\tilde{\phi}_{\infty}(b) = m_p = (i-1)\pi_p + \pi_p$ if $k$ is odd. If we compare with
function of the \(i\)-th eigenfunction with boundary condition (5) if \(k\) is even and (5\(ab\)) is \(k\) is odd. This identifies \(v_\infty\) as \(\lambda_j^b\) if \(k\) is even and \(\lambda_j^{ab}\) if \(k\) is odd.

In the singular case \(a = 0\) and \(x \geq p - 1\) the argument is very similar. This time the first local minimum \(\tilde{r}_1 \to 0 = \tilde{R}_1\). Thus the boundary condition on the first nodal interval \(\tilde{R}_1, R_2 = [0, R_2]\) is \(\phi_\omega(0+) = 3\pi p / 2\) and \(\phi_\omega(2) = 2\pi p\). With the same subtraction of multiples of \(\pi p\) as before, cf. (30), we obtain a smooth function \(\phi_\omega\) of (29) with \(\phi_\omega(0) = \pi p / 2\), \(\phi_\omega(b) = (i - 1) \pi p + \delta\) if \(k\) is even, \(\phi_\omega(b) = i \pi p + \delta\) if \(k\) is odd. This identifies \(v_\infty\) as \(\lambda_j\) if \(k\) is even and as \(\lambda_j^b\) if \(k\) is odd.

The limit \(v \to \infty\). The proof is like the previous one. We only sketch the argument. Now the interior zeroes condense in the following form: \(R_k = (k\) is even, \(b\) is odd) and \(k\) (\(k\) is odd) and \(b\) (\(k\) odd) and \(\phi_\omega\) is a \(C_1\)-solution of (31). Thus \(\lambda_\omega = \lambda_j^b\) if \(k\) is even and \(\omega_\omega = \lambda_j^{ab}\) if \(k\) is odd.

7. PROOF OF THEOREM 3

Our proof is by a shooting argument. First we consider the case where \((\lambda_1, 0) \in \sigma_1^+, (\mu_{k+1}, r_k) \in \sigma_k^+\) and \((\mu_{k+1}, r_{k+1}) \in \sigma_{k+1}^+\). For \(r \in \mathbb{R} \setminus \{0\}\) let \(u(r, \tau)\) be the solution of \(L_p u + f(r, u) = 0\) with initial values

Case (S): \(u(0) = 0, u'(0) = 0\),

Case (R): \(u(a) = \tau, (Bu)(a) = u_0, \text{ if } \gamma_2 \neq 0, u'(a) = \tau, (Bu)(a) = u_0, \text{ if } \gamma_2 = 0\).

The function \(v(r, \tau) = u(r, \tau)/\tau\) has the initial values

Case (S): \(v(0) = 1, v'(0) = 0\),

Case (R): \(v(a) = 1, (Bu)(a) = u_0/\tau^{(p-1)}, \text{ if } \gamma_2 \neq 0, v'(a) = 1, (Bu)(a) = u_0/\tau^{(p-1)}, \text{ if } \gamma_2 = 0\).
and satisfies
\[ L^*_p v + f(r, \tau v) \tau^{(p-1)} = L^*_p w + m_+(v^+)^{p-1} - m_-(v^-)^{p-1} = 0 \quad \text{in } I, \]
where \( m_+(r, \tau) = f(r, \tau v) \tau^{(p-1)} \) if \( v(r, \tau) > 0 \) and \( m_+ = 0 \) otherwise. The asymptotic conditions on the nonlinearity \( f \) imply \( |f(r, t)| \leq A |t|^{p-1} + B \) for some constants \( A, B > 0 \). Thus, by Corollary 1, \( |u(\cdot, \tau)|_C^\ell \leq \text{const.} |\tau| \) and \( |v(\cdot, \tau)|_C^\ell \) is bounded uniformly in \( \tau \). Moreover, the asymptotic conditions on \( f \) are equivalent to the following two conditions
\[-\infty < -K - o(1) \leq \frac{f(r, t)}{r^{(p-1)}} \leq \lambda_1 s + q - \kappa + o(1) \]
or
\[ \begin{align*}
(\mu_k s + q + \kappa) - o(1) &\leq \frac{f(r, t)}{r^{(p-1)}} \leq (\mu_{k+1} s + q + \kappa) - o(1), & t > 0, \\
(\nu_k s + q + \kappa) - o(1) &\leq \frac{f(r, t)}{r^{(p-1)}} \leq (\nu_{k+1} s + q + \kappa) - o(1), & t < 0,
\end{align*} \]
where \( o(1) \to 0 \) tends to 0 uniformly for \( r \in I \) as \( |t| \to \infty \). By the uniform boundedness of \( v(\cdot, \tau) \) this implies
\[-\infty < -K - o(1) \leq m_+(r, \tau) \leq \lambda_1 s + q - \kappa + o(1) \]
or
\[ \begin{align*}
(\mu_k s + q + \kappa) - o(1) &\leq m_+(r, \tau) \leq (\mu_{k+1} s + q + \kappa) + o(1), \\
(\nu_k s + q + \kappa) - o(1) &\leq m_-(r, \tau) \leq (\nu_{k+1} s + q + \kappa) + o(1).
\end{align*} \]
Moreover, since \( m_+(r, \tau) \) are uniformly bounded in \( L^\infty \), there exists sequences \( |\tau_k| \to \infty \) such that \( m_+(\cdot, \tau_k) \) converges weak* to \( m_+ \in L^\infty(I) \). Since
\[ r^\gamma v'(r, \tau_k)^{(p-1)} - (r^\gamma v'(r, \tau_k)^{(p-1)})'(a) \]
\[ = - \int_a^r p^\gamma [m_+(p, \tau_k)(v(p, \tau_k)^+)^{p-1} - m_-(p, \tau_k)(v(p, \tau_k)^-)^{p-1}] \, dp \]
and since \( v(\cdot, \tau_k) \) and \( r^\gamma v'(\cdot, \tau_k)^{(p-1)} \) are equicontinuous we may take the limit \( k \to \infty \) and find that \( v(\cdot, \tau_k) \) converges uniformly to a solution \( v \) of
\[ L_p v + m_+(v^+)^{p-1} - m_-(v^-)^{p-1} = 0 \quad \text{in } I \]
with initial values

Case (S): \( v(0) = 1, \quad v'(0) = 0, \)
Case (R): \( v(a) = 1, \quad (Bv)(a) = 0, \quad \text{if } \gamma_2 \neq 0, \)
\( v'(a) = 1, \quad (Bv)(a) = 0, \quad \text{if } \gamma_2 = 0, \)
where $m_\pm$ satisfies the inequalities of $m_\pm(\cdot, \tau)$ but without the $o(1)$-term. If $\phi$ and $\phi_{k, +1}$ denote the argument functions for $\tau$ and the Fučík-eigenfunctions $u_{\mu_k} \tau_k$, $u_{\mu_{k+1}} \tau_{k+1}$, then Theorem 4 yields

$$0 < \phi(b) < \phi_1(b) = \delta$$

or

$$\delta + (k - 1) \pi_p = \phi_k(b) < \phi(b) < \phi_{k+1}(b) = \delta + k \pi_p,$$

i.e., $(Bu)(b) \neq 0$ and in particular $(Bu(\cdot, \tau_k))(b) \neq 0$ for $|\tau_k|$ large. Recalling that $u(x, \tau_k) = u(x, \tau_k) \tau_k$ we see that either $(Bu(\cdot, \tau_k))(b) \to +\infty$ as $\tau_k \to +\infty$ and $(Bu(\cdot, \tau_k))(b) \to -\infty$ as $\tau_k \to -\infty$ or vice versa. By continuous dependence of $u(\cdot, \tau)$ on $\tau$ there exists a $\tilde{\tau}$ with $(Bu(\cdot, \tilde{\tau}))(b) = u_1$, which proves the existence of a solution to (1), (2), (3). If $(0, \tilde{\lambda}_1) \in \sigma_1^\pi$, $(\mu_{k+1}, \tau_{k+1}) \in \sigma_{k+1}$, the proof works with the same shooting argument if we choose the initial values for $u$ with $-\tau$ instead of $\tau$.]

REFERENCES

