Convergence of the Neumann Laplacian on shrinking domains. (English summary)

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*Analysis (Munich)* 21 (2001), no. 2, 171–204.

The author considers a family of Neumann Laplacians $H_{\Omega_{\varepsilon}}$, $0 < \varepsilon \leq 1$, on domains $\Omega_{\varepsilon}$ of $\mathbb{R}^2$ such that $\{\Omega_{\varepsilon}\}$ shrinks to a tree $\Gamma$ in the sense that
\[
\begin{align*}
\Omega = \Omega_1 \supset \Omega_{\varepsilon_2} \supset \Omega_{\varepsilon_1} \quad (1 > \varepsilon_2 > \varepsilon_1), \\
\lim_{\varepsilon \to 0} \Omega_{\varepsilon} &= \Gamma.
\end{align*}
\]

The main question of this paper is in what sense the operator $H_{\Omega_{\varepsilon}}$ converges to the corresponding differential operator $H_{\Gamma}$ on the tree $\Gamma$.

Throughout this work, the author assumes that there exists a map $\tau$ from the domain $\Omega$ to the tree $\Gamma$ which is locally Lipschitz continuous except on a null set. Using this map, it is possible to introduce two weighted Hilbert spaces $L^2(\Gamma)$ and $H^1(\Gamma)$. Then the “Neumann Laplacian” $H_\Gamma$ in $L^2(\Gamma)$ is defined from a natural sesquilinear form on $H^1(\Gamma)$. Similarly, for each $\varepsilon \in (0, 1]$, starting with the subdomain $\Omega_{\varepsilon}$ and a map $\tau^{(\varepsilon)}$ from $\Omega_{\varepsilon}$ to the tree $\Gamma$, the “Neumann Laplacian” $H^{(\varepsilon)}_\Gamma$ in $L^2(\Gamma)$ is also defined.

Under some additional conditions on each $\tau^{(\varepsilon)}$ it is shown that, for any family of approximating subdomains $\Omega_{\varepsilon}$, there exists a subsequence $\{\Omega_{\varepsilon_n}\}_{n=1}^\infty$, $\varepsilon_n \downarrow 0$ ($n \to \infty$) of $\{\Omega_{\varepsilon}\}_{0 < \varepsilon \leq 1}$ such that the weak limits
\[
\tilde{w} - \lim_{n \to \infty} Q^{(\varepsilon_n)}_{\mu}(H_{\Omega_{\varepsilon_n}} - z)^{-1}(f_{\varepsilon_n}) = \tilde{w} - \lim_{n \to \infty} (H^{(\varepsilon_n)}_\Gamma - z)^{-1}(\gamma f)
\]
exist in $L^2(\Gamma)$, where $Q^{(\varepsilon_n)}_{\mu}$ is a bounded linear operator from $L^2(\Omega_{\varepsilon})$ into $L^2(\Gamma)$; $f \in H^1(\Omega)$ and $f_{\varepsilon}$ is the restriction of $f$; $\gamma$ is the trace operator from $H^1(\Omega)$ into the weighted $L^2$ space on the tree $\Gamma$; and $z \in \mathbb{C} \setminus [0, \infty)$.

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