Szász, Otto

On Möbius’ inversion formula and closed sets of functions.


Let $1 = \lambda_1, \lambda_2, \cdots$ be a sequence such that the product of two $\lambda$’s is again a $\lambda$. The author studies the transformation (1) $y(t) = T x(t) = \sum a_n x(\lambda_n t)$. Conditions are given under which (1) has an inverse of the form (2) $x(t) = \sum b_n y(\lambda_n t)$. The simplest of these is: in the space of bounded functions (1) has a unique inverse (2) if

$$\sum_{n=2}^{\infty} |a_n| < |a_1|.$$  

The work is based on the formal identity

$$\sum a_n \lambda^{-s} \sum b_m \lambda_m^{-s} = 1.$$  

A particular study is made of the case that $\{a_n\}$ and $\{\lambda_n\}$ are completely multiplicative, i.e., $a_m a_n = a_{mn}$, $\lambda_m \lambda_n = \lambda_{mn}$. In this case the unique formal inverse (2) is given by $b_n = \mu(n) a_n$, where $\mu(n)$ is the Möbius function.

If the series (1) and (2) are uniformly convergent, then the functions $\{x_n = x(\lambda_n t)\}$ and $\{y_n = y(\lambda_n t)\}$ determine the same closed linear manifold. Several applications of this remark are given, of which the following are typical. Every even continuous function vanishing at $t = 0$ can be uniformly approximated by finite linear combinations of the functions $\varphi(nt)$, where $\varphi(t)$ is even and of period 2 and, in $0 \leq t \leq 1$, $\varphi(t) = \sin \frac{1}{2} \pi t$ or $\varphi(t) = t$.

The completeness of $\{y(\lambda_n t)\}$ in $L_r$ ($1 \leq r \leq \infty$) is discussed under various hypotheses on $x(t)$. In particular, $\{y(\lambda_n t)\}$ is complete in $L_r(a,b)$ if $\{x(\lambda_n t)\}$ is complete in $L_r(a,b)$ and if (i) $\sum a_n x(\lambda_m \lambda_n t)$ can be integrated term by term and (ii) $\sum a_n \xi(\lambda_m \lambda_n) = 0$ for $m = 1, 2, \cdots$ implies $\xi(\lambda_m) = 0$ ($m = 1, 2, \cdots$). Many applications are given. Examples are: $\{nt - [nt] - \frac{1}{2}\}$ is complete in $L_r(0, \frac{1}{2})$ for $r > 1$; $\{\text{sgn} \sin n\pi t\}$ is complete in $L_r(0, 1)$ for $r > 1$.

Every function continuous in $< 0, 1 >$ can be uniformly approximated by linear combinations of $\{1, t, t^n (\log t^{-1})^\delta\}$ ($\delta > 0$). The same is true of $\{1, t, (\log 1/t)^{1+\beta} k_\beta(t)\}$ ($-1 < \beta \leq 2$, $k_\beta = \sum n^\beta t^n$). The methods and results of this paper show many points of similarity with those of D. G. Bourgin [same Trans. 60, 478–518 (1946); MR0020168 (8,512f)].

Reviewed by W. H. J. Fuchs

© Copyright American Mathematical Society 1948, 2006