Research Plan - 2001

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My work so far falls into three or four broad categories, all of which are associated with function spaces, harmonic analysis, weighted inequalities and spectral theory.

I plan to continue working on multipliers and integral operator, as well as to pursue my recent research about approximation numbers into spectrum distribution of differential operators on irregular domains and to study the properties of the p-Laplacian. I am also interested in weighted inequalities and their applications to partial differential equations, and to obtain sharp estimates for the approximation and the entropy numbers for the Sobolev embedding and to evaluate the best constant for the Poincare inequality.

I. Approximation numbers and eigenvalues estimates on irregular domains.

The spectral properties of the Neumann and Dirichlet Laplacian $\Delta$ in $L^2(\Omega)$, where $\Omega$ is a domain in $\mathbb{R}^n$, have been studied for a long time. It is well known that the spectrum of $\Delta$ is discrete when $\Omega$ is bounded and has either a $C^1$ boundary or the $W^{1,2}$-extension property, and it is also well know that

$$\lim_{\lambda \to \infty} N(\lambda)\lambda^{-n/2} = \frac{1}{(2\pi)^n}\omega_n|\Omega|$$

where $N(\lambda)$ counts the number of the eigenvalues of the negative Neumann (or Dirichlet) Laplacian which are $\leq \lambda$, (See H. Weyl [41]).

When $\Omega$ has irregular boundary, (e.g. fractal boundary, horns, ...) the distribution of the eigenvalues of $\Delta$ is more difficult to understand. For example, the reminder term in the asymptotic formula for $N(\lambda)$ may depend on the Minkovski dimension of the boundary, (see M. Lapidus, [33] or G. Metivier, [36]). When $\Omega$ has not the $W^{1,2}$-extension property the usual direct methods for obtaining the asymptotic of $N(\lambda)$ do not work.

The spectrum properties of the Neumann Laplacian $\Delta_N$ in $L^2(\Omega)$ are determined by the nature of the embedding map $E : W^{1,2}(\Omega) \to L^2(\Omega)$ and by the approximation numbers of $E$, which are defined by

$$a_n(E) = \inf\{\|E - P_n\|; P_n : W^{1,2}(\Omega) \to L^2(\Omega), \text{Dim(Rank}(P_n) < n\}.$$ 

When $E$ is a compact embedding, then $a_n(E) = \mu_n(E)$ where $\mu_n(E)$ is the $m$-th singular value of $E$, i.e. the $m$-th eigenvalue of the positive, compact, self-adjoint operator $(EE^*)^{1/2}$ in $L^2(\Omega)$. In this case we have

$$a_n(E) = \lambda_n^{-2}$$

where $\lambda_n$ is $m$-th eigenvalue of $-\Delta_N$ in $L^2(\Omega)$. 
Therefore, we can estimate the eigenvalues of the Neumann Laplacian by estimating the approximation numbers of the Sobolev embedding theorem.

For domain with $W^{1,p}$ extension property, two-sided bounds were obtained by E. Kollo [26] for the Kolmogorov numbers of $E$, which coincide with the approximation numbers when when $p = 2$, (see [10, Sect. II]).

A very good technology for studying the approximation numbers for the Sobolev embedding on irregular domains without the $W^{1,2}$-extension property was developed by W.D. Evans, and D.J. Harris, (see [17] and [18]), provided that it is possible to find a tree inside the domain for which all the "bad" subsets of the boundary are at the ends of vertices of a tree. Such domains are called generalized ridge domains. For example, the majority of the domains with fractal boundaries are generalized ridge domains.

In this case it was showed that the problem of estimating the approximation numbers of the Sobolev embedding can be related to the problem of estimating the approximation numbers for the Hardy type operators on intervals or on trees, (see e.g. [18] for domains which have only one caps and [19] for fractal domains like the Koch snowflake). A Hardy type operator on a interval or on a tree is an operator of the form of

$$Tf(x) := v(x) \int_a^x f(t)u(t)dt, \quad x \in \Gamma$$

where $\Gamma$ is a tree (or a interval) and $a \in \Gamma$ is a root of that tree (or one endpoint of the ends of the interval).

In [11] and [12] the authors studied these Hardy type operators on intervals and found an upper and lower estimate for the approximation numbers of $a_n(T)$. In [12], (which was inspired by J. Newman and M. Solomyak [38]), it was shown that

$$\lim_{n \to \infty} na_n(T) = \frac{1}{\pi} \int_I vu,$$

where $I$ is an interval.

If we regard $T$ as an operator from $L^p(I)$ to $L^p(I)$, with $p \in (1, \infty)$ and is $\neq 2$, then they only show that $\lim \sup na_n(T)$ and $\lim \inf na_n(T)n$ are bounded above and below by a constant times $\int uv$. In [20] W.D. Evans, D.J. Harris and myself considered the cases $p = 1$ and $p = \infty$. If $\Gamma$ is a tree we proved in [21] that

$$\lim_{n \to \infty} na_n(T) = \int_\Gamma vu$$

for any $1 < p < \infty$. In this paper we develop a new and original technique which, on trees, can provide new results which are better that the know results on intervals. We also estimated $\|a_n(T)\|_{L^p(I)}$.

In [37] K. Naimark and M. Solomyak studied independently the problem for trees in the case $p = 2$, and their results are covered by our result from [21].
In my recent joint paper with D.E. Edmunds and R. Kerman [13] we improved the known estimates for the approximation numbers on intervals for \( p = 2 \). We obtain
\[
a_n(T) = \frac{1}{\pi n} \int_I |u(t)v(t)|dt + O(n^{-3/2}),
\]
which, to the best of my knowledge, the best known result, (compare with [12] or [37]).

In the next few years I would like to:

- Extend the results from [13] to case \( p \neq 2 \) and to trees, and then use these results and the results from [21] to improve the existing techniques and to obtain better estimates for the approximation numbers of the Sobolev embedding and the eigenvalues of the Neumann Laplacian on irregular domains.

- Use the results from [21] to obtain information about \( \|a_n(E)\|_{l^{p,q}} \) and \( \|\lambda_n\|_{l^{p,q}} \) on irregular domains.

- Study the relationship between the changes in the domain and the changes in the \( a_n(T) \)'s and the distribution of the \( \lambda_n \)'s from the Neumann Laplacian, (these problems are more difficult for the the Neumann Laplacian than for the Dirichlet Laplacian since the \( \lambda_n \)'s are not monotonic).

- Study the properties of the Sobolev embedding when \( E \) is non compact. This case corresponds, (for \( p = 2 \)) to the case in which the essential spectrum of \( \Delta_N \) is non-empty. I would like to get understand the properties of essential spectrum of \( \Delta_N \) on domains with irregular boundaries, (e.g., ”rooms” and ”passages”), which play an important role in quantum mechanics.

II. The Sobolev embedding and the Poincare inequality

Let \( I \) be the Sobolev embedding
\[
I : W^{1,p}(Q) \to L^q(Q)
\]
where \( Q = (0,1)^n \subset \mathbb{R}^n \) and \( 1 \leq p \leq q \leq \frac{np}{n-p} \). It is well known (see for example B.Carl [7], M.S. Birman and M.Z. Solomyak [2],[5] and H. Triebel [39]) that entropy numbers of \( I \) satisfy:
\[
c_1m^{-1/n} \leq e_m(I) \leq c_2m^{-1/n}.
\]
For the definition of entropy numbers see [10].

For the approximation numbers of \( I \) the situation is different. It is know that:
\[
c_3m^\beta \leq a_m(I) \leq c_4m^\beta
\]
where \( \beta < 0 \) is a number which depends on \( p, q, n \). This is true for any choice of \( p, q, n \) except for the case in which \( 1 \leq p \leq 2 \leq q \leq \infty \) and \( 1 \leq n + \max(1-1/q, 1/q) \). In this case
the exact rate of decay of $a_m(I)$ it is unknown. See H. Triebel [39],[40] or D.E. Edmunds and H. Triebel [16] for details.

Unfortunately it is know that $\lim_{m \to \infty} m^{-1/n} e_m(I)$ and $\lim_{m \to \infty} m^{\beta} a_m(I)$ exist only when $p = q = 2$. In this case the $a_m(I)$’s and $e_m(I)$’s are related to the eigenvalues of the Neumann Laplacian and thus we can use the spectral theory. When $p$ or $q$ are different from 2 it is not clear whether the above limits exist or not.

In [15] D.E. Edmunds and myself proved that if $1 \leq p = q \leq \infty$ and $n = 1$, then $\lim_{m \to \infty} m^{\beta} a_m(I)$ exists. To the best of my knowledge, that is the only known result when either $p$ or $q$ are different from 2.

In [15] we use the results of my joint papers [21] and [13], and we also use results of P. Lindquist [34] and P. Drabek and R. Manasevich [9] about $p$-Laplacian and the Poincare inequality.

In my joint papers [29] and [24] I have developed techniques which may help me to obtain similar results in 2 or higher dimensions, provided that I can find the best constant $C$ for the Poincare inequality

$$\|\nabla u\|_{L^q(Q)} \leq C\|u - u_Q\|_{L^p(Q)}$$

when $n \geq 1$. In the next few years I would like to focus on the following problems:

- Find the best constant and the extremal functions for the above Poincare inequality when $n > 1$.
- Use the best constant and the extremal function for the Poincare inequality in dimension when $n > 1$ to compute the precise asymptotics for the approximation numbers of the Sobolev embedding.
- Use the techniques of D.E. Edmunds and myself to obtain precise asymptotic for the entropy numbers of the Sobolev embedding, (the problem has been posed by H. Triebel)
- Apply these results to the study of the distribution of the eigenvalues of the $p$-Laplacian in higher dimensions and to the study of non-linear differential equations.

III. Convolution and integral operators.

In the excellent paper of [6] D.W. Boyd gave a full characterization of the rearrangement invariant spaces in which is the Hilbert transform $Hf(x) = p.v.\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt$ is bounded. Another important convolution operator which is related to the Hilbert transform is the segment multiplier

$$Sf(x) = \int_{-\infty}^{\infty} \frac{\sin(x - y)}{x - y} f(y) dy,$$

which play an important rule in physics applications.

In a recent paper by L. De Carli and E. Laeng [8] it is shown that the $L^p(\mathbb{R}) - L^p(\mathbb{R})$ operator norms of $H$ and $S$ coincide for all $1 < p < \infty$. This result may suggest that
$H$ is bounded if and only if $S$ is bounded, but this is not true in general rearrangement invariant spaces. In [24] R. Kerman and myself constructed an Orlicz space on which $H$ is unbounded and $S$ is bounded. We also introduced a new scale of indices on rearrangement invariant spaces in terms of which we can provide necessary and sufficient conditions for the boundedness of $S$. I am currently completing a joint paper with R. Kerman [25] in which we provide a full characterization of the boundness of $S$ on weighted Sobolev spaces.

- I would like to use the technique from my joint papers [24] and [25] to study the boundness of other convolution operators with oscillating kernels (like, for example, kernels which contain Bessel functions), and then I would like to generalize these result to rearrangement invariant spaces and weighted spaces in dimension 2 or higher.

Another integral operator in which I am interested is the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dx.$$ 

Necessary and sufficient condition for the boundedness and the compactness of the Riesz potential in weighted $L^p$ spaces (the trace inequality)

$$\|I_\alpha f\|_{L^q(\omega)} \leq c\|f\|_{L^p}$$

can be found in many classical textbooks, (see e.g. [1]).

Unfortunately these condition are quite hard to check. In the paper [35] and [23] by I. Verbitsky and V.G. Mazya, and by N. Kalton and I. Verbitsky, the authors provide new and easy to check necessary and sufficient conditions for the boundedness of the above Riesz potential.

- I would like to use these new conditions for obtaining simple and easy to check conditions for the compactness of the Riesz potential.

IV. Banach function spaces.

The largest possible scale of spaces in which we can study general integral operators are the Banach function spaces. These spaces contains the Lebesgue, Lorentz and Orlicz spaces etc. In my joint papers [32] and [22] I studied the properties of the Hardy-type integral operators acting on Banach function spaces. In [30] A. Nekvinda, L. Pick and myself provided a complete characterization of the compact integral operators which maps Banach function space into $L^\infty$. We have also been able to measure the non compactness of the Hardy type operators, that is, the distance between these operators and the subspace of the compact operators, by using the continuity of the norms of the weights of the Hardy type operators. The notion of continuous norm was first introduced by L. Pick and Q. Lai in [27], and then studied in [31] and [28].

- In the next few years I would like to continue the investigation of the properties of the integral operators mapping Banach function spaces into $BMO$ and $VMO$ and to
measure the non-compactness of such operators. That will help to improve the Sobolev embedding into $BMO$ and $VMO$ (see [14]).
References


