Estimates for the Approximation numbers and n-widths of the weighted Hardy-type operators

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Abstract

The weighted Hardy-type integral operator $T : L^p(a, b) \to L^p(a, b)$, $-\infty \leq a < b \leq \infty$, is defined by

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)dt.$$ 

In papers by Edmunds, Evans & Harris (EEH1) and Edmunds, Harris & Lang (EHL1) upper and lower estimates and asymptotic results were obtained for the Approximation numbers $a_n(T)$ of $T$. In the case $p = 2$ for “nice” $u$ and $v$ these results were improved in Edmunds, Kerman & Lang (EKL) and lately extended for $1 < p < \infty$ in Lang (L). In this paper we will improve these results and obtain the second asymptotic term and also extend these results for Kolmogorov, Gel’fand and Bernstein numbers.

Key words: Approximation, Kolmogorov, Gel’fand and Bernstein numbers, weighted Hardy-type operators, Integral operators, Weighted Spaces

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1 Introduction.

For the weighted Hardy operator $T$ defined by

$$(T f)(x) = v(x) \int_0^x u(t) f(t) dt,$$  \hspace{1cm} (1)

and being a map from $L^p(a, b)$ into $L^p(a, b)$, for $1 \leq p \leq \infty$, properties of the Approximation numbers were studied in (EEH1), (EEH2), (LL) and (EHL1). From papers (NS2), (NS1) and (EHL2), under some conditions on $u$ and $v$, it was shown that the Approximation numbers $a_n(T)$ of $T$ in the case $1 < p < \infty$ satisfy

$$\lim_{n \to \infty} na_n(T) = \alpha_p \int_a^b |u(t)v(t)| dt,$$

where $\alpha_p = (1/\lambda_p)^{1/p}$ ($\lambda_p$ corresponds to the first eigenvalue of the $p$-Laplacian problem on interval $(0, 1)$ and $\lambda_p = \left( \frac{2\pi}{p^{p-1} \sin(\pi/p)} \right)$ (see (EL)). From this it follows that

$$\frac{1}{C} \leq \lim_{n \to \infty} \left( a_n(T) - \frac{\alpha_p}{n} \|uv\|_{1,(a,b)} \right) \leq C, \quad \text{for some } C > 0.$$

Under slightly stricter conditions on weights $u, v$ these results were improved in (EKL) (case $p = 2$) and later in (L) (case $1 < p < \infty$). It was shown that

$$\limsup_{n \to \infty} n^{1/2} \left| \alpha_p \int_a^b |u(t)v(t)| dt - na_n(T) \right| \leq c(\|u\|_{p/(p'+1)} + \|v\|_{p/(p'+1)}) (\|u\|_{p'} + \|v\|_p) + 3\alpha_p \|uv\|_1.$$  \hspace{1cm} (2)

In this paper techniques from (L) are improved and by using information about properties of $A(I)$ (introduced in (EHL1)) we obtain information about the first asymptotic for $n$-widths and also information about the second asymptotic for the Approximation numbers and $n$-widths numbers for the weighted Hardy operator. Mainly we proved:

$$\rho_n(T) = \frac{1}{n} \alpha_p \int_I u(x)v(x) dx + O(n^{-2})$$
where \( \rho_n(T) \) stands for any of the following: the Approximation numbers of \( T \), Kolmogorov, Gel’fand or Bernstein n-widths of \( T \).

We mention here that in the case \( u = v = 1 \) (i.e. non-weighted case) the problem of description of Approximation numbers and n-widths for the non-weighted Hardy operator and corresponding Sobolev embedding was already studied and described in (M), (BMN), (TB), (EL) and (L1).

Also we would like to bring to the reader’s attention a recent elegant paper (B) in which similar results were obtained by using different techniques.

## 2 Asymptotic estimate for the Approximation and n-widths numbers.

We start by recalling the definitions of the Approximation numbers and n-widths.

**Definition 2.1** Let \( T : L^p(I) \to L^p(I) \) be a bounded operator and \( n \in \mathbb{N} \).

(i) The n-th approximation number \( a_n(T) \) of \( T \) is defined by

\[
a_n(T) := \inf \| T - F | L^p(T) \to L^p(I) \|
\]

where the infimum is taken over all bounded linear maps \( F : L^p(I) \to L^p(I) \) with rank less than \( n \).

(ii) The n-th Kolmogorov width \( d_n(T) \) of \( T \) is defined by

\[
d_n(T) = d_n(T(L^p(I)), L^p(I)) = \inf_{X_n} \sup_{\| x \|_{L^p(I)} \leq 1} \inf_{y \in X_n} \| Tx - y \|_{L^p(I)}
\]

where the infimum is taken over all \( n \)-dimensional subspaces \( X_n \) of \( L^p(I) \).

(iii) The n-th width in the sense of Gel’fand \( d^n(T) \) of \( T \) is defined by

\[
d^n(T) = d^n(T(L^p(I)), L^p(I)) = \inf_{L^n} \sup_{\| x \|_{L^p(I)} \leq 1, x \in L^n} \| Tx \|_{L^p(I)}
\]

where the infimum is taken over all \( n \)-dimensional subspaces \( X_n \) of \( L^p(I) \).

(iv) The Bernstein n-th width \( b_n(T) \) of \( T \) is defined by

\[
b_n(T) = b_n(T(L^p(I)), L^p(I)) = \sup_{X_{n+1}} \inf_{T x \in X_{n+1}, T x \neq 0} \| Tx \|_{L^p(I)} / \| x \|_{L^p(I)}
\]

where \( X_{n+1} \) is any subspace of span \( \{ T x ; x \in X \} \) of dimension \( \geq n + 1 \).

(v) The linear n-th width \( \delta_n(T) \) of \( T \) is defined by

\[
\delta_n(T) = \delta_n(T(L^p(I)), L^p(I)) = \inf_{P_n} \sup_{\| x \|_{L^p(I)} \leq 1} \| Tx - P_n x \|_{L^p(I)}
\]
where $P_n$ is any continuous linear operator of $L^p(I)$ into $L^p(I)$ of rank at most $n$.

The following lemma will give us information about relation between the Approximation numbers and $n$-widths.

**Lemma 2.2** Let $T : L^p(I) \in L^p(I)$ be a bounded operator and $n \in \mathbb{N}$, then

$$a_{n+1}(T) = \delta_n(T) \geq d_n(T), d^n(T) \geq b_n(T).$$

**Proof.** The first equality is obvious. For the rest, see (P) □

Throughout the paper we shall assume that $-\infty \leq a < b \leq \infty$ and that

$$u \in L^{p'}(a, b), \; v \in L^p(a, b) \quad \text{and} \quad u, v > 0 \text{ on } (a, b). \quad (3)$$

Under these restrictions on $u$ and $v$ it is well known (see, for example, (EEH1), Theorem 1) that the norm $\|T\|$ of the operator $T : L^p(a, b) \to L^p(a, b)$ in (1) satisfies

$$\|T\| \sim \sup_{x \in (a, b)} \|u \chi_{(a,x)}\|_{p'},(a,b) \|v \chi_{(x,b)}\|_{p,(a,b)}. \quad (4)$$

Here $\chi_S$ denotes the characteristic function of the set $S$ and

$$\|f\|_{p,I} = \left( \int_I |f(t)|^p dt \right)^{1/p}, \; 1 < p < \infty, \; I \subset (a, b).$$

Moreover, by $F_1 \sim F_2$ we mean that $C^{-1}F_1 \leq F_2 \leq CF_1$ for some positive constant $C \geq 1$ is independent of any variables in $F_1, F_2 \geq 0$.

From (3) it also follows that the operator $T$ is a compact operator from $L^p$ into $L^p$ (see (EGP) or (OK)).

Next we introduce a function $A$ which will play a key role in the paper. Given $I = (c, d) \subset (a, b)$, set

$$A(I, u, v) := \sup_{\|f\|_{p,I} = 1} \inf_{\alpha \in \mathbb{R}} \|Tf - \alpha v\|_{p,I}. \quad (we \; will \; write \; A(I) \; in \; a \; situation \; in \; which \; is \; obvious \; which \; functions \; are \; u, v.)$$

Since $T$ is a compact operator then from (EHL2), Theorem 3.8. we have

$$A(I, u, v) = \inf_{x \in I} \|T_{x,I}L^p(I) \to L^p(I)\|,$$
where

\[ T_{x,I}f(.) := v(.)\chi_{I}(.) \int_{x} v(t)\chi_{I}(t)dt. \]

**Lemma 2.3** Let \( I = (c, d) \subset (a, b) \) and \( 1 \leq p \leq \infty \), then \( \|T_{x,I}\|_{L^p(I) \to L^p(I)} \) is continuous in \( x \).

**Proof.** See Lemma 3.4 in (EHL2) and put \( \Gamma = (a, b) \) and \( K = I \). \( \Box \)

**Lemma 2.4** Suppose that \( u \) and \( v \) satisfy (3), \( a \leq c < d \leq b \) and \( 1 < p < \infty \). Then:

1. The function \( A(.,d) \) is non-increasing and continuous on \( (a,d) \).
2. The function \( A(c,.) \) is non-decreasing and continuous on \( (c,b) \).
3. \( \lim_{y \to a^+} A(a,y) = \lim_{y \to b^-} A(y,b) = 0 \).

**Proof.** See Lemma 2.2 in (L). \( \Box \)

From the previous two lemmas, we can obtain the next lemma.

**Lemma 2.5** Suppose that \( T : L^p(a,b) \to L^p(a,b) \) is compact and \( 1 < p < \infty \). Let \( I = (c,d) \) and \( J = (c',d') \) be subintervals of \( (a,b) \), with \( J \subset I \), \( |J| > 0 \), \( |I - J| > 0 \), \( \int_{a}^{b} v^p(x)dx < \infty \) and \( u,v > 0 \) on \( I \). Then

\[ A(I) > A(J) > 0. \]

(5)

**Proof.** See Lemma 2.3 in (L). \( \Box \)

Now we will introduce \( N(\varepsilon) \) which plays an important rule.

**Remark 2.6** It follows from the continuity of \( A \) that for sufficiently small \( \varepsilon > 0 \) there is an \( a_1, a < a_1 < b \), for which \( A(a_1,b) = \varepsilon \). Indeed, since \( T \) is compact, there exists a positive integer \( N(\varepsilon) \) and points \( b = a_0 > a_1 > \ldots > a_{N(\varepsilon)} = a \) with \( A(a_i, a_{i-1}) = \varepsilon, i = 1,2,\ldots,N(\varepsilon) - 1 \) and \( A(a,a_{N(\varepsilon)-1}) \leq \varepsilon \).

With the same arguments as in the proof of Lemma 2.6 from (EKL) we obtain from the previous two lemmas the following lemma:

**Lemma 2.7** If \( T : L^p(a,b) \to L^p(a,b) \) is compact and \( v \in L^p(a,b) \), \( u \in L^{p'}(a,b) \) then the number \( N(\varepsilon) \) is a non-increasing function of \( \varepsilon \) which takes on every sufficiently large an integer value.

The quantity \( N(\varepsilon) \) is useful in the derivation of upper and lower estimates for the Approximation numbers of \( T \) as we can see from the following lemma,
which is an easy consequence of Lemma 3.19 from (EHL2) (put $K = (a, b)$).

**Lemma 2.8** For all $\varepsilon \in (0, \|T\|)$,

$$a_{N(\varepsilon)+2}(T) \leq \varepsilon \leq a_{N(\varepsilon)-1}(T).$$

Next, we extend this lemma also for n-widths.

**Lemma 2.9** Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a, b)$. If $N := N(\varepsilon)$ then

$$a_{N+1}(T) \leq \varepsilon.$$

**Proof.** Let us recall that $T$ is compact and then from Lemma 2.7 it follows that for any $\varepsilon > 0$ we have $N(\varepsilon) < \infty$. Let $\{I_i\}_{i=1}^N$ be the partition of $I$ which defines $N := N(\varepsilon)$ in Remark 2.6 and set $P f = \sum_{i=1}^N P_i f$ where

$$P_i f(x) := \chi_{I_i} v(x) \left[ \int_a^{a_i} f u + \int_{c_i}^{a_i} f u \chi_{I_i} \right],$$

and $a_i$ is the left end point of $I_i$, $c_i$ is the point from $I_i$ such that $A(I_i) = \|T_{c_i,I_i}|L^p(I) \rightarrow L^p(I)\|$ (for existence of such a point see Lemma 3.14 in (EHL2)).

Then $\text{rank}(P) \leq N$ and we have

$$\|(T - P)f\|_{p,I}^p = \sum_{i=1}^N \|T f - P f\|_{p,I_i}^p$$

$$= \sum_{i=1}^N \|\chi_{I_i} v(.) \int_a^{a_i} f(t) dt - P_i f(.)\|_{p,I_i}^p$$

$$= \sum_{i=1}^N \|\chi_{I_i} v(.) \int_{c_i}^{a_i} f(t) dt\|_{p,I_i}^p$$

$$\leq \sum_{i=1}^N (A(I_i))^p \|f\|_{p,I_i}^p$$

$$= \left( \max_{i=1,...,N} A(I_i) \right)^p \|f\|_{p,I}^p$$

and then the lemma follows. \(\Box\)

**Lemma 2.10** Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a, b)$. If $N := N(\varepsilon)$ then

$$b_{N-2}(T) \geq \varepsilon.$$
Proof. From the definition of $N(\varepsilon)$ we have that for $i = 1, \ldots, N-1$, $A(I_i) = \varepsilon$. Let $\lambda \in (0, 1)$; then from the definition of $A(I_i)$ we have, for each $i = 1, \ldots, N-1$, a function $\phi_i \in L^p(I)$, where $\|\phi_i\|_{p,I} = 1$, with support in $I_i$ such that

$$\inf_{\alpha \in \mathbb{R}} \|T\phi_i - \alpha v\|_{p,I} > \lambda A(I_i) \geq \lambda \varepsilon.$$ 

Let $X_{N-1} = \text{span}\{T\phi; \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i, \lambda_i \in \mathbb{R}\}$. Then we can see that rank $X_{N-1} \geq N - 1$. Take $0 \neq T\phi \in X_{N-1}$ then $0 \neq \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i$ with $\lambda_i \neq 0$ for some $i$.

$$\|T\phi\|_{p,I}^p \geq \sum_{i=1}^{N-1} \| (T\phi) \chi_{I_i} \|_{p,I}^p = \sum_{i=1}^{N-1} \| \chi_{I_i}(x)v(x) \left( \int_a^x \lambda_i \phi_i(t) \chi_{I_i}(t) dt + \int_a^x \phi(t)u(t) dt \right) \|_{p,I}^p$$

$$= \sum_{i=1}^{N-1} \left\| \left( T\phi_i(x) + v(x) \frac{\eta_i}{\lambda_i} \right) \lambda_i \right\|_{p,I}^p \left( \text{where } \eta_i := \int_a^x \phi(t)u(t) dt \right)$$

$$\geq \sum_{i=1}^{N-1} \inf_{\alpha \in \mathbb{R}} \|T\phi_i(x) - v(x)\alpha\|_{p,I_i}^p |\lambda_i|^p$$

$$\geq (\lambda \varepsilon)^p \sum_{i=1}^{N-1} \|\phi_i\|_{p,I_i}^p |\lambda_i|^p \geq (\lambda \varepsilon)^p \|\phi\|_{p,I_i}^p,$$

and the lemma follows. \(\square\)

The next theorem follows from these lemmas.

**Theorem 2.11** Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a, b)$. If $N := N(\varepsilon)$ then

$$a_{N+1}(T) \leq \varepsilon \leq b_{N-2}(T).$$

We can see that this theorem gives us

$$a_{N+1}(T) \leq \varepsilon \leq a_{N-1}(T) \quad \text{and} \quad \rho_N(T) \leq \varepsilon \leq \rho_{N-2}(T)$$

where $\rho_N(T)$ stands for any of the following $\delta_N(T), d_N(T), d^N(T)$ or $b_N(T)$.

For general $u$ and $v$ it is impossible to find a simple relation between $\varepsilon$ and $N(\varepsilon)$, but by using the properties of $A(I)$ the behavior of $\varepsilon N(\varepsilon)$ when $\varepsilon \to 0_+$
can be determined.

**Lemma 2.12** Given \( v \in L^p(a, b), u \in L^{p'}(a, b) \) then we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon N(\varepsilon) = \alpha_p \int_a^b |u(t)v(t)|dt.
\]

This result follows from an adaptation of the argument of (EHL2); see, in particular, Theorem 6.4 of that paper. Together with Theorem 2.11 this shows, again using the techniques of (EHL2), that the following theorem holds.

**Theorem 2.13** Given \( v \in L^p(a, b), u \in L^{p'}(a, b) \) the operator \( T \) defined in (1) satisfies

\[
\lim_{n \to \infty} n \rho_n(T) = \alpha_p \int_a^b |u(t)v(t)|dt,
\]

where \( \alpha_p = A((0, 1), 1, 1) \) and \( \rho_n(T) \) stands for any of the followings \( b_n(T), \delta_n(T), d_n(T), d_n(T) \) or \( a_n(T) \).

From paper (L) we have better estimate for the relation between \( \varepsilon > 0 \) and \( N(\varepsilon) \) (see Theorem 4.1 in (L))

**Lemma 2.14** Let \( -\infty \leq a < b \leq \infty \) and \( I = (a, b) \), let \( u \in L^{p'}(I), v \in L^p(I) \) and suppose that \( u' \in L^{p/(p+1)}(a, b) \cap C([a, b]), v' \in L^{p/(p+1)}(a, b) \cap C([a, b]). \) Then

\[
\limsup_{\varepsilon \to 0^+} |\alpha_p \int_a^b |u(t)v(t)|dt - \varepsilon N(\varepsilon)|N^{1/2}(\varepsilon) \leq c(p, p') \left( \|u''\|_{p/(p'+1),I} + \|v''\|_{p/(p+1),I} \right) \left( \|u\|_{p',I} + \|v\|_{p,(a,b)} \right) + 3\alpha_p\|uv\|_{1,I},
\]

where \( \alpha_p = A((0, 1), 1, 1) \) and \( c(p, p') \) is a constant depending only on \( p \) and \( p' \).

From this Lemma and from Theorem 2.11 we can easily obtain the following theorem:

**Theorem 2.15** Let \( -\infty \leq a < b \leq \infty \) and \( I = (a, b) \), let \( u \in L^{p'}(I), v \in L^p(I) \) and suppose that \( u' \in L^{p/(p+1)}(a, b) \cap C([a, b]), v' \in L^{p/(p+1)}(a, b) \cap C([a, b]). \)
C([a, b]). Then

\[ \limsup_{n \to \infty} |\alpha_p \int_{a}^{b} |u(t)v(t)dt - \rho_n(T) n^{1/2} \leq \]

\[ c(p, p') \left( \|u'\|_{p'_/(p'+1), I} + \|v'\|_{p/(p+1), I} \right) \left( \|u\|_{p', I} + \|v\|_{p, (a,b)} \right) + 3\alpha_p \|uv\|_{1, I}, \]

where \( \alpha_p = A((0, 1), 1, 1) \), \( c(p, p') \) is a constant depending only on \( p \) and \( p' \) and \( \rho_n(T) \) stands for any of the followings \( \delta_n(T), d_n(T), d'_n(T), b_n(T) \) or \( a_n(T) \).

3 The second asymptotic term

In this section we will use properties of \( A(I) \) to obtain better estimate about the Approximation numbers and \( n \)-widths numbers.

At first some observation about \( A(I) \).

**Lemma 3.1** Let \( I = (c, d) \subseteq (a, b) \) and \( d = (c + d)/2 \). Suppose that \( u \) and \( v \) are constant functions over \( I \). Then

\[ A(I, u, v) = |I| \|u\| |v| A((0, 1), 1, 1) \]

and

\[ \sup_{f \in L^p(I)} \inf_{c \in \mathbb{R}} \frac{\|v(x) \int_{a}^{x} u(t)f(t)dt - c\|_{p, I}}{\|f\|_{p, I}} = \]

\[ \sup_{f \in L^p(I)} \frac{\|v(x) \int_{a}^{x} u(t)f(t)dt - c\|_{p, I}}{\|f\|_{p, I}} = \]

\[ = |u||v| \left( \frac{\sin(\pi p/(2))}{\pi^{p-1}} \right) \left( \frac{1}{\lambda_p} \right)^{1/p} |u||v|, \]

where \( \lambda_p = \frac{1}{\alpha_p} = \left( \frac{2\pi}{\sin(\pi/2)} \right) \frac{1}{p'(p-1)} \) is the first non-zero eigenvalue of the \( p \)-Laplacian problem on interval \((0, 1)\).

**Proof.** See Lemma 4.1 in (EHL2) and Lemma 2.7 in (EL). \( \square \)
In this lemma $\sin_p(.)$ and $\cos_p(.)$ mean special goniometric functions, which correspond to first non-constant eigenfunction of the one-dimensional p-Laplacian (see (EL) or (DM) for more).

From the definition of $A(I)$ we have:

**Lemma 3.2** Let $I = (c, d) \subset (a, b)$. Let $u_1 \geq u_2 > 0$ and $v_1 \geq v_2 > 0$ than we have:

$$A(I, u_1, v_1) \geq A(I, u_2, v_2) \geq 0.$$ 

Now we are ready to prove the following lemma about behavior of $\varepsilon N(\varepsilon)$.

**Lemma 3.3** Let $1 < p < \infty$, $I = (a, b)$, $u \in L^p(I)$, $v \in L^p(I)$ and $(v'/v), (u'/u) \in L^1(I) \cap C[a, b]$ then

$$\lim_{\varepsilon \to 0^+} \left| N(\varepsilon) \left[ \varepsilon N(\varepsilon) - \alpha_p \int_I u(x)v(x)dx \right] \right|$$

$$\leq \int_I u(x)v(x)dx \left[ \int_I \frac{v'(x)}{v(x)} dx + \int_I \frac{u'(x)}{u(x)} dx + \alpha_p \right]$$

$$+ \left( \int_I \frac{u'(x)}{u(x)} dx \right) \left( \int_I \frac{v'(x)}{v(x)} dx \right).$$

**Proof.** Take $||T|| > \varepsilon > 0$ and $N := N(\varepsilon)$. Then we have the following partition: $I = \bigcup_{i=1}^N A(I_i) = \varepsilon$ for $i = \{1, ..., N - 1\}$ and $A(I_N) < \varepsilon$. Define the following step functions:

$$u^{+\varepsilon}(x) = \sum_{i=1}^N u_i^{+\varepsilon} \chi_{I_i}(x), \quad \quad v^{+\varepsilon}(x) = \sum_{i=1}^N v_i^{+\varepsilon} \chi_{I_i}(x)$$

$$u^{-\varepsilon}(x) = \sum_{i=1}^N u_i^{-\varepsilon} \chi_{I_i}(x), \quad \quad v^{-\varepsilon}(x) = \sum_{i=1}^N v_i^{-\varepsilon} \chi_{I_i}(x)$$

where

$$u_i^{+\varepsilon} = \sup_{x \in I_i} |u(x)|, \quad \quad u_i^{-\varepsilon} = \inf_{x \in I_i} |u(x)|$$

$$v_i^{+\varepsilon} = \sup_{x \in I_i} |v(x)|, \quad \quad v_i^{-\varepsilon} = \inf_{x \in I_i} |v(x)|.$$
Then we have from the previous two lemmas:

$$\alpha_p u_i^- \varepsilon v_i^- |I_i| \leq A(I_i) \leq \alpha_p u_i^+ \varepsilon v_i^+ |I_i|,$$

and we can see that

$$\int_{I} u^- \varepsilon(x)v^- \varepsilon(x)dx \leq \int_{I} u(x)v(x)dx \leq \int_{I} u^+ \varepsilon(x)v^+ \varepsilon(x)dx.$$

Let us estimate the upper bound for the following quantity:

$$K(\varepsilon) := \int_{I} (u^+ \varepsilon(x)v^+ \varepsilon(x) - u^- \varepsilon(x)v^- \varepsilon(x))dx$$

$$= \sum_{i=1}^{N} |I_i|(u_i^+ \varepsilon v_i^+ \varepsilon - u_i^- \varepsilon v_i^- \varepsilon)$$

$$= \sum_{i=1}^{N} |I_i|(u_i^+ \varepsilon v_i^+ \varepsilon - u_i^- \varepsilon v_i^- \varepsilon + u_i^+ \varepsilon v_i^- \varepsilon - u_i^- \varepsilon v_i^- \varepsilon)$$

(\text{use: } (v_i^+ \varepsilon - v_i^- \varepsilon) \leq |I_i| \max_{x \in I_i} |v'(x)|

and \((u_i^+ \varepsilon - u_i^- \varepsilon) \leq |I_i| \max_{x \in I_i} |u'(x)|\))

$$\leq \sum_{i=1}^{N} |I_i| \left[ u_i^+ \varepsilon |I_i| \max_{x \in I_i} |v'(x)| + v_i^- \varepsilon |I_i| \max_{x \in I_i} |u'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[ \frac{u_i^+ \varepsilon}{u_i^- \varepsilon} |I_i| \max_{x \in I_i} |u'(x)| + \frac{v_i^- \varepsilon}{v_i^- \varepsilon} |I_i| \max_{x \in I_i} |u'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[ |I_i| \max_{x \in I_i} |u'(x)| + |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$+ \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[ |I_i| \max_{x \in I_i} |u'(x)| + |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$+ \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$+ \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$+ \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$\leq \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

$$+ \frac{\varepsilon}{\alpha_p} N \left[ 1 + |I_i| \max_{x \in I_i} |u'(x)| \right] \left[ |I_i| \max_{x \in I_i} |v'(x)| \right]$$

11
\[
= \epsilon \frac{\alpha_p}{\sum_{i=1}^{N} \left| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{+\varepsilon}} \right| + \varepsilon \frac{\alpha_p}{\sum_{i=1}^{N} \left| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-\varepsilon}} \right|} \\
+ \epsilon \alpha_p \left( \sum_{i=1}^{N} \left| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-\varepsilon}} \right| \right) \left( \sum_{i=1}^{N} \left| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-\varepsilon}} \right| \right). 
\]

From (6) we have:

\[\sum_{i=1}^{N} \alpha_p u_i^{-\varepsilon} v_i^{-\varepsilon} |I_i| \leq \varepsilon N \quad \text{and} \quad \sum_{i=1}^{N} \alpha_p u_i^{+\varepsilon} v_i^{+\varepsilon} |I_i| \geq \varepsilon (N - 1)\]

and then

\[\sum_{i=1}^{N} \alpha_p u_i^{-\varepsilon} v_i^{-\varepsilon} |I_i| - \alpha_p \int_{I} uv \, dx \leq \varepsilon N - \alpha_p \int_{I} uv \, dx \]

\[\leq \sum_{i=1}^{N} \alpha_p u_i^{+\varepsilon} v_i^{+\varepsilon} |I_i| + \varepsilon - \alpha_p \int_{I} uv \, dx\]

which gives us

\[-K(\varepsilon) \leq \varepsilon N - \alpha_p \int_{I} uv \, dx \leq K(\varepsilon) + \varepsilon\]

and

\[-NK(\varepsilon) \leq N \left( \varepsilon N - \alpha_p \int_{I} uv \, dx \right) \leq NK(\varepsilon) + \varepsilon N.\]

Using \(\lim_{\varepsilon \to 0^+} (\varepsilon N(\varepsilon)) = \alpha_p \int_{I} uv \, dx\) and that

\[\lim_{\varepsilon \to 0^+} \frac{K(\varepsilon)}{\varepsilon} = \frac{1}{\alpha_p} \int_{I} \frac{u'}{u} + \frac{1}{\alpha_p} \int_{I} \frac{v'}{v} + \frac{1}{\alpha_p} \int_{I} \frac{u'}{u} \int_{I} \frac{v'}{v}\]

we obtain:

\[\limsup_{\varepsilon \to 0^+} \left| N \left( \varepsilon N - \alpha_p \int_{I} uv \, dx \right) \right| \leq \left( \int_{I} uv \right) \left( \int_{I} \frac{u'}{u} + \int_{I} \frac{v'}{v} + \int_{I} \frac{u'}{u} \int_{I} \frac{v'}{v} \right) + \alpha_p \int_{I} uv. \square\]
With help of Lemma 2.2 we obtain from the previous lemma the following theorem

**Theorem 3.4** Let \(-\infty < a < b \leq \infty\) and \(I = (a,b)\), let \(u \in L^p(I)\), \(v \in L^p(I)\) and \((v'/v), (u'/u) \in L^1(I) \cap C[a,b]\) then

\[
\limsup_{n \to \infty} \left| n \left[ n\rho_n(T) - \alpha_p \int_I u(x)v(x) \, dx \right] \right|
\]

\[
\leq \int_I u(x)v(x) \, dx \left[ \int_I \frac{v'(x)}{v(x)} \, dx + \int_I \frac{u'(x)}{u(x)} \, dx + 2\alpha_p \right]
\]

\[
+ \left( \int_I \frac{u'(x)}{u(x)} \, dx \right) \left( \int_I \frac{v'(x)}{v(x)} \, dx \right) ,
\]

where \(\rho_n(T)\) stands for any of the followings: \(a_n(T)\), \(\delta_n(T)\), \(d_n(T)\), \(d^n(T)\) or \(b_n(T)\), and \(T\) is the Hardy-type operator.

**Proof.** From Lemma 2.7 we have that for any large \(n\) there is \(\varepsilon > 0\) such \(n = N(\varepsilon)\). Then from Theorem 2.11 we have:

\[
n \left( \varepsilon n - \alpha_p \int_I uv \right) \geq n \left( a_{n+1}(T) n - \alpha_p \int_I uv \right)
\]

\[
\geq [(n+1) - 1] \left( a_{n+1}(T) [(n+1) - 1] - \alpha_p \int_I uv \right)
\]

\[
\geq (n+1) \left( a_{n+1}(T) (n+1) - \alpha_p \int_I uv \right)
\]

\[
+ (n+1)(-a_{n+1}(T)) - \left( a_{n+1}(T) n - \alpha_p \int_I uv \right)
\]

and

\[
n \left( \varepsilon n - \alpha_p \int_I uv \right) \leq n \left( a_{n-1}(T) n - \alpha_p \int_I uv \right)
\]

\[
\leq [(n - 1) + 1] \left( a_{n-1}(T) [(n - 1) + 1] - \alpha_p \int_I uv \right)
\]

\[
\leq (n - 1) \left( a_{n-1}(T) (n - 1) - \alpha_p \int_I uv \right)
\]

13
\[(n - 1)a_{n-1}(T) + \left( a_{n-1}(T)n - \alpha \int uv \right).\]

By taking the limit \( n \to \infty \) and with the help of Theorem 2.13 we proved our theorem for \( \rho_n(T) = a_n(T) \) and by using similar technique we can get the proof of the theorem for \( d_n(T), d^n(T), b_n(T) \) and \( \delta_n(T) \).

From Theorem 3.4 we have the following information about the second asymptotic:

\[
\rho_n(T) = \frac{1}{n} \alpha \int u(x)v(x)dx + O(n^{-2}).
\]

**Remark 3.5** We have found that our method which is based on studying of behavior \( \varepsilon N(\varepsilon) \) cannot be improved behind the second term.

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