Complex Analysis

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1. Complex numbers and functions

1.1. The field of complex numbers

1.1.1. The set \( \mathbb{C} \) of complex numbers is the 2-dimensional \( \mathbb{R} \)-vector space with basis \( \{1, i\} \), so that complex numbers are expressions of the form \( x + iy \), \( x, y \in \mathbb{R} \). Multiplication in \( \mathbb{C} \) is defined by the formula

\[
(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)
\]

or, assuming the distributive law, simply by the identity \( i^2 = -1 \). With respect to the addition in \( \mathbb{R}^2 \) and this multiplication, \( \mathbb{C} \) is a field, where for \( z = x + iy \neq 0 \) one has \( z^{-1} = \frac{x - iy}{x^2 + y^2} \). The punctured complex plane \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) is a group under multiplication.

1.1.2. For \( z = x + iy \in \mathbb{C} \), the coordinate \( x \) is called the real part of \( z \) and is denoted by \( \text{Re} \, z \), and the coordinate \( y \) is called the imaginary part of \( z \) and is denoted by \( \text{Im} \, z \). The complex number \( i = 1i \) is called the imaginary unit.

Real numbers are identified with complex numbers of the form \( x + 0i = x \), which makes \( \mathbb{R} \) a subfield of \( \mathbb{C} \). The numbers of the form \( 0 + iy \) are called imaginary.

1.1.3. For \( z = x + iy \in \mathbb{C} \), the number \( \bar{z} = x - iy \) is called the conjugate of \( z \). The operation of conjugation preserves addition and multiplication in \( \mathbb{C} \): for any \( z_1, z_2 \in \mathbb{C} \), \( \overline{z_1 + z_2} = \bar{z_1} + \bar{z_2} \) and \( \overline{z_1 z_2} = \bar{z_1} \bar{z_2} \). (So, conjugation is an automorphism of \( \mathbb{C} \).)

For every \( z \in \mathbb{C} \) we have \( z + \bar{z} = 2 \text{Re} \, z \) and \( z - \bar{z} = 2i \text{Im} \, z \).

1.1.4. For \( z = x + iy \in \mathbb{C} \), the absolute value, or the modulus of \( z \) is the nonnegative real number \( |z| = \sqrt{x^2 + y^2} \).

The absolute value function has the following properties:

(i) \(|0| = 0 \) and \(|z| > 0 \) for all \( z \neq 0 \);
(ii) for any \( z \in \mathbb{C} \), \(|z|^2 = z \bar{z} \);
(iii) multiplicativity: for any \( z_1, z_2 \in \mathbb{C} \), \(|z_1 z_2| = |z_1| |z_2| \);
(iv) the triangle inequality: for any \( z_1, z_2 \in \mathbb{C} \), \(|z_1 + z_2| \leq |z_1| + |z_2| \);
(v) for any \( z \in \mathbb{C} \), \(|\text{Re} \, z|, |\text{Im} \, z| \leq |z| \leq |\text{Re} \, z| + |\text{Im} \, z| \).

1.1.5. The metric on \( \mathbb{C} \) is defined by \( \text{dist}(z_1, z_2) = |z_1 - z_2| \), which converts \( \mathbb{C} \) into a metric space. In this metric, the open balls are discs: the disc of radius \( r \) centered at \( z_0 \in \mathbb{C} \) is \( \Delta(z_0, r) = \{ z : |z - z_0| < r \} \). The boundary of \( \Delta(z_0, r) \) is the circle \( K(z_0, r) = \{ z : |z - z_0| = r \} \).

1.1.6. The unit circle \( K(0,1) \) is the set of complex numbers of absolute value 1: \( K(0,1) = \{ w \in \mathbb{C} : |w| = 1 \} \). For any \( w_1, w_2 \in K(0,1) \) we have \( w_1 w_2, w_1^{-1} \in K(0,1) \).

1.1.7. For every \( z \neq 0, |z|/z \in K(0,1) \), so \( z = |z|w \) where \( w \in K(0,1) \), and so, \( w = \cos \theta + i \sin \theta; \theta \) is called the argument of \( z \) and is denoted by \( \text{arg} \, z \).

1.1.8. \( \text{arg} \, z \) is not defined uniquely: if \( \theta = \text{arg} \, z \), then also \( \theta + 2k \pi = \text{arg} \, z \) for all \( k \in \mathbb{Z} \). \( \text{arg} \) is an example of a multivalued functions, that may take several values at the points of its domain: for a multivalued function \( F \) and for \( z \in \text{Dom} \, F \), \( F(z) \) is a set, not a single element.

If \( F \) is a multivalued function and \( D \subset \text{Dom} \, F \), a branch of \( F \) on \( D \) is a continuous function on \( D \) such that \( f(z) \in F(z) \) for all \( z \in D \).
1.1.9. **The principal value** \( \text{Arg } z \) of the argument of \( z \in \mathbb{C}^* \) is the element \( \theta \) of \( \text{arg } z \) satisfying \(-\pi < \theta \leq \pi\). \( \text{Arg} \) is a branch of \( \text{arg} \) on \( \mathbb{C} \setminus (-\infty, 0] \).

1.1.10. For \( z_1, z_2 \in \mathbb{C} \), \( \text{arg}(z_1z_2) = \text{arg } z_1 + \text{arg } z_2 \).

1.2. **Algebraic functions**

The graph of a function from a subset of \( \mathbb{C} \) to \( \mathbb{C} \) is, normally, a two-dimensional surface in a four-dimensional \( \mathbb{R} \)-vector space, and we cannot draw it. Instead, we consider such functions as transformations of subsets of \( \mathbb{C} \).

1.2.1. The function \( f(z) = z + c \) for \( c \in \mathbb{C} \) is a shift of \( \mathbb{C} \) by \( c \).

1.2.2. The function \( f(z) = cz \) for a positive \( c \in \mathbb{R} \) is the stretch of \( \mathbb{C} \) by a factor of \( c \). For \( c \in \mathbb{C}^* \), the function \( f(z) = cz \) is the stretch of \( \mathbb{C} \) by a factor of \(|c|\) and the rotation about 0 by the angle of \( \text{arg } c \): \(|cz| = |c||z|\) and \( \text{arg}(cz) = \text{arg } z + \text{arg } c \).

1.2.3. The function \( f(z) = z^{-1} \) is the composition of the reflection of \( \mathbb{C} \) with respect to \( K(0,1) \) and the reflection with respect to \( \mathbb{R} \).

1.2.4. The function \( f(z) = z^n \) maps a point \( z \) with \(|z| = r \) and \( \text{arg } z = \theta \) to the point \( z^n \) with \(|z^n| = r^n \) and \( \text{arg } z^n = n \text{arg } z \). It performs an \( n \)-fold covering of \( \mathbb{C} \); each point of \( \mathbb{C} \) except zero has exactly \( n \) preimages.

1.2.5. The inverse of \( z^n \), \( \sqrt[n]{z} \), is a multivalued function: each point of \( \mathbb{C} \) except zero has exactly \( n \) images. The graph of this function consists of \( n \) “branches” – \( n \) copies of \( \mathbb{C} \) covering \( \mathbb{C} \), which are “cut and glued together”, so that after making a loop about zero one passes from one branch to the other. (0 is said to be a **ramification point** for this mapping.)

1.2.6. **Zhukovsky’s function** \( f(z) = \frac{1}{2}(z + z^{-1}) \) is a 2-to-1 function on \( \mathbb{C}^* \) (except for the points \( \pm 1 \)). It maps the unit circle \( K(0,1) \) onto the interval \([-1,1] \), both the disc \( \{ z : |z| < 1 \} \) and the annulus \( \{ z : |z| > 1 \} \) onto \( \mathbb{C} \setminus [-1,1] \).

1.3. **The exponential and the logarithmic functions**

1.3.1. The complex **exponential function** \( \exp z = e^z \) is defined as \( e^{x+iy} = e^x (\cos y + i \sin y) \), that is, \(|e^z| = e^x\) and \( \text{arg } e^z = y \). \( \exp \) maps \( \mathbb{C} \) onto \( \mathbb{C}^* \), and is \( 2\pi i \)-periodic: \( e^{z+2\pi i} = e^z \) for all \( z \in \mathbb{C} \).

1.3.2. For any \( z_1, z_2 \in \mathbb{C} \), \( e^{z_1+z_2} = e^{z_1}e^{z_2} \).

1.3.3. By \( \exp \), the vertical lines \( \{ z : \text{Re } z = a \} \) in \( \mathbb{C} \) are transformed to the circles \( K_{0,e^a} = \{ w : |w| = e^a \} \), and the horizontal lines \( \{ z : \text{Im } z = b \} \) are transformed to the rays \( \{ w : \text{arg } w = b \} \).

The vertical strips \( \{ z : a_1 < \text{Re } z < a_2 \} \) are transformed to the annuli \( \{ w : e^{a_1} < |w| < e^{a_2} \} \), and the horizontal strips \( \{ z : b_1 < \text{Im } z < b_2 \} \) are transformed to the sectors \( \{ w : \text{arg } w < b_2 \} \). For every \( k \in \mathbb{Z} \), \( \exp \) maps the strip \( \{ z : (2k-1)\pi \leq \text{Re } z < (2k+1)\pi \} \) onto \( \mathbb{C}^* \) bijectively.

1.3.4. The points of the unit circle \( K(0,1) \) are exactly the numbers of the form \( e^{it} = \cos t + i \sin t \) with \( t \in \mathbb{R} \). Thus, every \( z \in \mathbb{C} \) is uniquely representable in the form \( z = |z|e^{it} \) (where \( t = \text{arg } z \)).
The inverse of exp is the logarithmic function \( \log \mathbb{C}^* \to \mathbb{C} \), defined by \( \log w = \log |w| + i \arg w \), that is, \( \log(re^{it}) = \log r + i t \). Since arg is a multivalued function, log is a multivalued function too: the imaginary part of \( \log z \) is defined up to \( 2\pi i \). The graph of \( \log \) (as well as the graph of arg) consists of infinite two-sided sequence of “branches” – copies of \( \mathbb{C} \) covering \( \mathbb{C} \), which are “cut and glued together”, so that after making a loop about zero one passes from one branch to the next one. (0 is a ramification point for this mapping.)

For any \( w_1, w_2 \in \mathbb{C}^* \), \( \log(w_1w_2) = \log w_1 + \log w_2 \). (However, \( \log(w_1w_2) \) may not be equal to \( \log w_1 + \log w_2 \).)

For \( z \in \mathbb{C}^* \) and \( w \in \mathbb{C} \), \( z^w \) is defined as \( e^{w \log z} \). This is, again, a multivalued function, that may have infinitely many values: for instance, \( i^i = e^{i(2k\pi i)} = e^{-2k\pi} \), for \( k \in \mathbb{Z} \).

The inverse of exp is \( \exp \) which takes infinitely many values at every point of \( \mathbb{C} \). The inverse of \( \exp \) is \( \log \).

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The logarithmic function \( \log \) is the function \( \log w = \log |w| + i \arg w \), that maps \( \mathbb{C}^* \) onto the set \( \{ z : -\pi < \Im z \leq \pi \} \); this is a branch of \( \log \) on the set \( \mathbb{C} \setminus (-\infty, 0] \).

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1.5. The Riemann sphere and Möbius transformations

1.5.1. The extended complex plane, or the Riemann sphere is the complex plane with the infinite point added, \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). Topologically, \( \hat{\mathbb{C}} \) is a sphere: via the stereographic projection (the projection from the north pole of the sphere to a plane orthogonal to the axis of the sphere through its poles), the points of the sphere are identified with the points of the plane, and the north pole of the sphere is assumed to be the point \( \infty \). Thus, in \( \hat{\mathbb{C}} \), a sequence \((z_n)\) converges to \( \infty \) if the sequence \(|z_n|\) converges to \( \infty \) in \( \mathbb{R} \).

1.5.2. The Möbius transformations, or the linear-fractional transformations, are functions on \( \mathbb{C} \) of the form \( \varphi(z) = \frac{az+b}{cz+d} \) with \( a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \). Such a mapping \( \varphi \) is not defined at the point \(-d/c\); we put \( \varphi(-d/c) = \infty \), and we put \( \varphi(\infty) = a/c \). After this, every Möbius transformation becomes a homeomorphism (a continuous bijections) \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). (We will later see that Möbius transformations are the only differentiable self-bijections of \( \hat{\mathbb{C}} \).

1.5.3. The composition of two Möbius transformations and the inverse of a Möbius transformation are Möbius transformations as well. (This means that Möbius transformations form a group.)

The Möbius transformation \( \varphi(z) = \frac{az+b}{cz+d} \) is defined by the matrix \( M_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with nonzero determinant; such a definition is not unique, since for any \( c \in \mathbb{C}^* \), the matrix \( cM_\varphi \) also defines \( \varphi \). The composition of two Möbius transformations is then defined by the product of the corresponding matrices, \( M_\varphi M_\psi = M_\varphi M_\psi \). (This means that we have a homomorphism from the group \( \text{GL}(2, \mathbb{C}) \) of nondegenerate \( 2 \times 2 \) complex matrices onto the group of Möbius transformations, and the group of Möbius transformations is therefore isomorphic to the factor group \( \text{GL}(2, \mathbb{C})/\mathbb{C}^* \).)

It also follows that the inverse of a Möbius transformation \( \varphi(z) = \frac{az+b}{cz+d} \) is the Möbius transformation \( \varphi^{-1}(w) = \frac{dw-b}{cw-a} \), defined by the matrix \( \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

1.5.4. Every Möbius transformation is representable as a composition of Möbius transformations of the form \( z \mapsto az+b \) with \( a \in \mathbb{C}^* \), \( z \mapsto z+b \) with \( b \in \mathbb{C} \), and \( z \mapsto z^{-1} \).

1.5.5. Every Möbius transformation \( \varphi \) has either one or two fixed points in \( \hat{\mathbb{C}} \) (that is, the points \( z \in \hat{\mathbb{C}} \) for which \( \varphi(z) = z \)). \( \infty \) is a fixed point of every Möbius transformation of the form \( \varphi(z) = az+b \), and is the only fixed point of transformations of the form \( \varphi(z) = z+b \).

1.5.6. Every Möbius transformation is uniquely defined by its action on any 3 distinct points: if \( \varphi_1(z_i) = \varphi_2(z_i), i = 1, 2, 3 \), for distinct points \( z_1, z_2, z_3 \in \hat{\mathbb{C}} \), then \( \varphi_1 = \varphi_2 \).

1.5.7. Möbius transformations are 3-transitive: for any 3 distinct points \( z_1, z_2, z_3 \in \hat{\mathbb{C}} \) and any three distinct point \( w_1, w_2, w_3 \in \hat{\mathbb{C}} \) there exists a (unique) Möbius transformation \( \varphi \) such that \( \varphi(z_i) = w_i, i = 1, 2, 3 \).

The Möbius transformation that maps \((z_1, z_2, z_3)\) to \((0, 1, \infty)\) respectively is \( \varphi(z) = \frac{z-z_2}{z_3-z_1} \cdot \frac{z-z_3}{z_2-z_1} \).

1.5.8. Möbius transformations preserve the cross ratio \( [z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} \); for any Möbius transformation \( \varphi \) and any \( z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}} \), \([\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)] = [z_1, z_2, z_3, z_4]\).

1.5.9. Let us call circles in \( \mathbb{C} \), as well as lines in \( \mathbb{C} \) with the point \( \infty \) added, circles in \( \hat{\mathbb{C}} \). (Every such “circle in \( \hat{\mathbb{C}} \)” is defined by an equation of the form \( p|z|^2 + qz + pr + r = 0 \) for some \( p, r \in \mathbb{R}, q \in \mathbb{C}, \) with \( |q|^2 > pr \). This is a straight line if \( p = 0 \).) Then every Möbius transformation transforms circles in \( \hat{\mathbb{C}} \) to circles in \( \hat{\mathbb{C}} \); moreover, every circle in \( \hat{\mathbb{C}} \) can be transformed to any other circle in \( \hat{\mathbb{C}} \).
1.5.10. Let us call open discs in \( \mathbb{C} \), open half-planes in \( \mathbb{C} \), as well as open complements in \( \mathbb{C} \) to closed discs in \( \mathbb{C} \) (that is, sets of the form \( \{ z : |z - z_0| > r \} \cup \{ \infty \} \) discs in \( \mathbb{C} \). Then every Möbius transformation transforms discs in \( \mathbb{C} \) to discs and \( \hat{\mathbb{C}} \), and moreover, every disc in \( \mathbb{C} \) can be transformed to any other disc in \( \mathbb{C} \).

1.5.11. Möbius transformations preserve angles between curves.

1.5.12. Anti-Möbius transformations are the mappings \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of the form \( \psi(z) = \frac{az + b}{c\bar{z} + d} \), that is, \( \phi(z) = \phi(\bar{z}) \) where \( \phi \) is a Möbius transformation. The composition of two anti-Möbius transformations is an Möbius transformation, and the composition of any anti-Möbius transformation and any Möbius transformation is an anti-Möbius transformation.

The reflection with respect to any line or circle in \( \mathbb{C} \) is an anti-Möbius transformation. (For \( K = K(z_0, r) \), the reflection with respect to \( K \) is the mapping \( \rho_K(z) = z_0 + \frac{r^2}{z - z_0} \).

1.5.13. Möbius transformations preserve symmetry with respect to circles in \( \hat{\mathbb{C}} \): for any Möbius transformation \( \phi \) and any circle \( K \) in \( \hat{\mathbb{C}} \), \( \phi \circ K = \rho_{\phi(K)} \).

1.5.14. The Möbius transformations that map the unit disc \( \Delta = \Delta(0, 1) \) to itself are those of the form \( e^{i\theta} \frac{az + b}{c\bar{z} + d} \) with \( \theta \in \mathbb{R} \) and \( z_0 \in \Delta \). (These Möbius transformations form a group.)

The Möbius transformations that map the upper half-plane \( \{ z : \Im z > 0 \} \) to itself are those of the form \( \frac{az + b}{c\bar{z} + d} \) with \( a, b, c, d \in \mathbb{R} \) and \( ad - bc > 0 \). (These Möbius transformations form a group.)

The Möbius transformation \( \phi(z) = i\frac{z + i}{z - 1} \) transforms \( \Delta \) to \( \hat{\mathbb{C}} \). The Möbius transformation \( \phi^{-1}(z) = \frac{z - 1}{z + 1} \) transforms \( \hat{\mathbb{C}} \) to \( \Delta \).

1.5.15. In Poincaré’s disc model of hyperbolic geometry “the plane” is the open disc \( \Delta = \Delta(0, 1) \), straight lines are the arcs of circles orthogonal to the disc’s boundary (and diameters of the disc are also permitted). In this geometry, there are infinitely many lines parallel to a given line and passing through a single point. The metric on “the plane” is defined by the formula \( \text{dist}(p_1, p_2) = \left| \log \left( \frac{(b_1 - p_2)(b_2 - p_1)}{(b_1 - p_1)(b_2 - p_2)} \right) \right| = \left| \log[b_1, p_1, p_2, b_2] \right| \), where \( b_1 \) and \( b_2 \) are the endpoints of “the line” (the arc) through the points \( p_1, p_2 \in \Delta \). The Möbius transformations of \( \Delta \), that is, of the form \( e^{i\theta} \frac{az + b}{c\bar{z} + d} \) with \( \theta \in \mathbb{R} \) and \( z_0 \in \Delta \), preserve this metric and constitute the group of motions of “the plane”; every point with a unit vector at it can be sent by such a Möbius transformation to any other point with any unit vector at it. The points of the boundary \( K(0, 1) \) of the disc are called the infinite points, the distance from these points to the points in “the plane” (that is, those inside the disc) is infinite.

A similar geometry can be constructed in the upper half-plane \( H = \{ z : \Im z > 0 \} \) (as well as in any other half-plane in \( \mathbb{C} \), of course).

1.5.16. Möbius transformations \( \phi_1 \) and \( \phi_2 \) are said to be conjugate if \( \phi_2 = \psi^{-1} \phi_1 \psi \) for some Möbius transformation \( \psi \). (If \( \psi \) is considered as a “change of coordinates” transformation of \( \mathbb{C} \), then the transformation \( \phi_2 \) can be viewed as the transformation \( \phi_1 \) written in different coordinates on \( \hat{\mathbb{C}} \).

Every Möbius transformation \( \phi \) is conjugate to (and so, looks like) exactly one of the following Möbius transformations:

(i) \( z \mapsto z + b \) (has a single fixed point and is called parabolic);
(ii) \( z \mapsto \pm z \) (circular);
(iii) \( z \mapsto az \) with \( |a| = 1, a \neq \pm 1 \) (elliptic);
(iv) \( z \mapsto az \) with \( a \in \mathbb{R}, a > 1 \) (hyperbolic);
(v) \( z \mapsto az \) with \( a \notin \mathbb{R}, |a| > 1 \) (loxodromic).
2. Differentiation

2.1. Complex differentiation

2.1.1. A linear function on \( \mathbb{C} \) is a function of the form \( z \mapsto cz + w \) for some \( c, w \in \mathbb{C} \).

Let \( c = a + bi \); then in “real” coordinates \( x, y \) this mapping has form \( (x, y) \mapsto C(z) + w \), where \( C = (a - b) \).

The matrix \( C \) can be written as \( C = |c|(\cos \theta - \sin \theta) \), where \( \theta = \arg c \), that is, the mapping \( z \mapsto cz + w \) is a composition of rotation (by \( \arg c \)) and translation (by \( |c| \)).

2.1.2. Let \( f \) be a function on a set \( S \subseteq \mathbb{C} \) and let \( z_0 \) be an interior point of \( S \). The limit if \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \) is called the derivative of \( f \) at \( z_0 \) and is denoted by \( f'(z_0) \); if this limit exists, \( f \) is said to be differentiable at \( z_0 \).

Equivalently, \( f \) is differentiable at \( z_0 \) if \( f \) “is approximable” by a linear function: \( f(z_0 + h) = f(z_0) + ch + o(h) \), in which case \( c = f'(z_0) \).

2.1.3. If a function \( f \) is differentiable at \( z_0 \), then \( f \) is continuous at \( z_0 \).

2.1.4. (i) For \( f = \text{const} \), \( f'(z_0) = 0 \) for every \( z_0 \in \mathbb{C} \).

(ii) If \( f \) and \( g \) are differentiable at \( z_0 \), then so are \( f + g \) and \( fg \), and \( (f + g)'(z_0) = f'(z_0) + g'(z_0) \), \( (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) \); if, additionally, \( g(z_0) \neq 0 \), then \( f/g \) is differentiable at \( z_0 \) and \( (f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2} \).

(iii) If \( f \) is differentiable at \( z_0 \) and \( g \) is differentiable at \( f(z_0) \), then \( g \circ f \) is differentiable at \( z_0 \) and \( (g \circ f)'(z_0) = g'(f(z_0))f'(z_0) \).

(iv) If \( f \) is differentiable at \( z_0 \), \( f'(z_0) \neq 0 \), and \( f^{-1} \) is continuous at \( f(z_0) \), then \( (f^{-1})'(f(z_0)) = 1/f'(z_0) \).

(v) For \( f(z) = z^n \), \( n \in \mathbb{N} \), \( f'(z_0) = nz_0^{n-1} \) for any \( z_0 \in \mathbb{C} \). For \( f(z) = z^{-n} \), \( n \in \mathbb{N} \), \( f'(z_0) = -nz_0^{-(n+1)} \) for any \( z_0 \in \mathbb{C}^* \).

(vi) If \( f \) is a branch of the function \( z^{1/n} \) on an open set \( U \), then \( f'(z_0) = \frac{1}{n}z_0^{(1/n)-1} \) for any \( z_0 \in U \).

(vi) \( (e^z)'|_{z_0} = e^{z_0} \), (any branch of) \( \log' z_0 = z_0^{-1} \) for any \( z_0 \in \mathbb{C}^* \).

(viii) \( \sin' z_0 = \cos z_0, \cos' z_0 = \sin z_0 \), (any branch of) \( \arcsin'(z_0) = \frac{1}{\sqrt{1-z_0^2}} \) for any \( z_0 \neq \pm 1 \), (any branch of) \( \arctan'(z_0) = \frac{1}{1+z_0} \) for any \( z_0 \neq \pm i \).

2.2. The Cauchy-Riemann equations

2.2.1. Let \( f:S \to \mathbb{C}, S \subseteq \mathbb{R}^2 \), be differentiable at an interior point \( z_0 \) of \( S \) “in the real sense”. The Cauchy-Riemann equation for \( f \) at \( z_0 \) is \( \frac{\partial f}{\partial x}(z_0) = -\frac{\partial f}{\partial y}(z_0) \), or in terms of \( u = \text{Re} f \) and \( v = \text{Im} f \), \( \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \) and \( \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0) \). In the matrix form, this means that the Jacobian \( f'(z_0) \) of \( f \) at \( z_0 \) has form \( \left( \begin{array}{c} a & -b \\ b & a \end{array} \right) \) for some \( a, b \in \mathbb{R} \), or, equivalently, \( |c| (\cos \theta \sin \theta) \) where \( c = a + bi \) and \( \theta = \arg c \).

2.2.2. Theorem. Let \( f:S \to \mathbb{C}, S \subseteq \mathbb{C} \), be defined in a neighborhood of a point \( z_0 \in S \) and differentiable at \( z_0 \) in the real sense. Then \( f \) is differentiable at \( z_0 \) in the complex sense iff it satisfies the Cauchy-Riemann equations at \( z_0 \).

2.2.3. If \( f:S \to \mathbb{C}, S \subseteq \mathbb{C} \), is differentiable in the real sense at a point \( z_0 \in S \), the partial derivatives \( \frac{\partial f}{\partial x}(z_0) \) and \( \frac{\partial f}{\partial y}(z_0) \) are defined as \( \frac{\partial f}{\partial x}(z_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right) \) and \( \frac{\partial f}{\partial z}(z_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0) \right) \). In these notations, \( f \) is differentiable at \( z_0 \) in the complex sense iff \( \frac{\partial f}{\partial z}(z_0) = 0 \).
2.3. Holomorphic functions

2.3.1. If a function \( f \) is differentiable at every point of an open set \( U \), then we say that \( f \) is holomorphic in \( U \). Functions holomorphic in entire \( \mathbb{C} \) are called entire functions.

The set of functions holomorphic in an open set \( U \) is denoted by \( \mathcal{O}(U) \).

2.3.2. If a function \( f \) is holomorphic in \( U \), then the function \( f'(z) \) is defined on \( U \) and is called the derivative function of \( f \).

The higher order derivatives of \( f \), if exist, are denoted by \( f^{(k)} \): \( f^{(0)} = f \) and \( f^{(k)} = (f^{(k-1)})' \) for \( k \in \mathbb{N} \).

2.3.3. If functions \( f \) and \( g \) are holomorphic in \( U \), then \( f + g \) and \( fg \) are holomorphic in \( U \), \( f/g \) is holomorphic in \( U \setminus \{ g = 0 \} \). (So, \( \mathcal{O}(U) \) is a ring.)

2.3.4. If \( f \) is holomorphic and \( f' = 0 \) in a domain \( D \), then \( f \) is constant on \( D \).

2.3.5. Proposition. If a function \( f \) holomorphic in a domain \( D \) is such that any of the functions \( |f| \), \( \text{Re } f \), or \( \text{Im } f \) is constant on \( D \), then \( f \) is constant on \( D \).

2.4. Holomorphic functions and conformal mappings

2.4.1. A differentiable mapping \( f: D \rightarrow \mathbb{R}^2 \) where \( D \) is a domain in \( \mathbb{R}^2 \) is said to be conformal if \( f \) preserves angles and the orientation; this is so iff for every \( z_0 \in D \), the derivative of \( f \) at \( z_0 \) is a composition of a stretch and a rotation. If \( f \) preserves angles but reverses the orientation, it is said to be anti-conformal.

2.4.2. Theorem. A function \( f: D \rightarrow \mathbb{C} \), where \( D \) is a domain in \( \mathbb{C} \), is conformal iff \( f \) is holomorphic in \( D \) and \( f'(z) \neq 0 \) for all \( z \in D \).

It follows that \( f \) is anti-conformal iff it is a composition of a holomorphic function with non-vanishing derivative and a reflection.

2.5. Primitive functions

2.5.1. A function \( F \) on an open set \( U \) is called a primitive of a function \( f \) on \( U \) if \( F' = f \) on \( U \).

2.5.2. If \( f \) has a primitive \( F \) on a domain \( D \), then \( G \) is a primitive of \( f \) iff \( G = F + \text{const} \).

2.5.3. The functions \( z^n \) for \( n \geq 0 \), \( e^z \), \( \sin z \), \( \cos z \) have primitives in \( \mathbb{C} \). The functions \( z^n \) for \( n \leq -2 \) have primitives in \( \mathbb{C}^* \). The function \( z^{-1} \) does not have a primitive in \( \mathbb{C}^* \), but has one in \( \mathbb{C} \setminus (-\infty, 0] \) (namely, \( \log z \)).

3. Contour integration

3.1. Integration along smooth paths

All open sets and all paths below are assumed to be in \( \mathbb{C} \).

3.1.1. Let \( \gamma: [a, b] \rightarrow \mathbb{C} \) be a path in \( \mathbb{C} \) and let \( f \) be a continuous function on \( |\gamma| \). It would be natural to define \( \int_{\gamma} f(z) \, dz \) as the limit, as mesh \( P \rightarrow 0 \), of the sums \( \sum_{j=1}^{n} f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1})) \) for partitions \( P = \{ a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \} \) of \( [a, b] \). Unfortunately, this definition may fail; it works well only for rectifiable paths, that is, paths of finite length. It is however established by Cauchy's theorem that in the case function \( f \) is holomorphic the integral \( \int_{\gamma} f \) does not change under deformations of the path; thus, after a slight deformation, we may assume that the path is smooth, and so rectifiable. Our plan is the following: we start by defining the integral along smooth paths, use this definition to prove Cauchy’s theorem, and then utilize this theorem to extend the definition of integral of a holomorphic function to general continuous paths.

3.1.2. We start with integration over real intervals. If \( f: [a, b] \rightarrow \mathbb{C} \) is a continuous function with \( u = \text{Re } f \) and \( v = \text{Im } f \), then \( \int_{a}^{b} f(t) \, dt \) is defined as \( \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt \).
3.1.3. If \( f: [a, b] \rightarrow \mathbb{C} \) is a continuous function and \( \varphi: [c, d] \rightarrow [a, b] \) is a continuously differentiable function, then \( \int_a^b f(t) \, dt = \int_c^d f(\varphi(s)) \varphi'(s) \, ds \).

3.1.4. If \( f: [a, b] \rightarrow \mathbb{C} \) is a continuous function with \( u = \text{Re} f \) and \( v = \text{Im} f \), then
\[
\left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt \leq \sup_{[a,b]} |f(t)| \cdot (b-a).
\]

3.1.5. A path \( \gamma: [a, b] \rightarrow \mathbb{C} \) is said to be smooth if \( \gamma \) is a continuously differentiable function (\( \gamma \) is differentiable and \( \gamma' \) is continuous). A path \( \gamma \) is said to be piecewise smooth if it is a finite sum \( \gamma = \gamma_1 + \cdots + \gamma_k \) of smooth paths.

3.1.6. Let \( \gamma: [a, b] \rightarrow \mathbb{C} \) be a smooth path and let \( f \) be a continuous function on \( |\gamma| \). Then the integral of \( f \) along \( \gamma \) is defined as
\[
\int_{\gamma} f(z) \, dz = \int_a^b f(\gamma(t))\gamma'(t) \, dt.
\]

The arclength integral of \( f \) along \( \gamma \) is defined as
\[
\int_{\gamma} |f(z)| \, dz = \int_a^b |f(\gamma(t))|\gamma'(t) \, dt.
\]

If \( \gamma \) is a piecewise smooth path, \( \gamma = \gamma_1 + \cdots + \gamma_k \) where \( \gamma_i \) are smooth, we define \( \int_{\gamma} f(z) \, dz = \sum_{i=1}^k \int_{\gamma_i} f(z) \, dz \) and \( \int_{\gamma} |f(z)| \, dz = \sum_{i=1}^k \int_{\gamma_i} |f(z)| \, dz \).

For integrals along closed paths there is a special notation, emphasizing the closeness of the path: \( \int_{\gamma} = \oint_{\gamma} \).

3.1.7. If \( \gamma \) is a constant path (\( |\gamma| \) is a single point), then for any \( f \), \( \int_{\gamma} f(z) \, dz = 0 \).

3.1.8. The operation of integration is linear with respect to functions: if \( \gamma \) is a piecewise smooth path and \( f, f_1, f_2 \) are functions continuous on \( |\gamma| \), then \( \int_{\gamma} (f_1 + f_2) \, dz = \int_{\gamma} f_1(z) \, dz + \int_{\gamma} f_2(z) \, dz \), and \( \int_{\gamma} cf(z) \, dz = c \int_{\gamma} f(z) \, dz \) for any \( c \in \mathbb{C} \).

3.1.9. Integration is additive with respect to paths: if \( \gamma \) is a sum of two piecewise smooth paths, \( \gamma = \gamma_1 + \gamma_2 \), then for every function \( f \) continuous on \( |\gamma| \), \( \int_{\gamma_1 + \gamma_2} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz \).

If \( \gamma \) is a piecewise smooth path, then for every function \( f \) continuous on \( |\gamma| \), \( \int_{-\gamma} f(z) \, dz = -\int_{\gamma} f(z) \, dz \).

3.1.10. The integral along a path \( \gamma \) doesn’t depend on the parametrization of \( \gamma \) (but depends on its direction): if \( \gamma \) is a piecewise smooth path and \( \tilde{\gamma} \) is obtained from \( \gamma \) by a piecewise smooth reparametrization, then for any function \( f \) continuous on \( |\gamma| \), \( \int_{\gamma} f(z) \, dz = \int_{\tilde{\gamma}} f(z) \, dz \). (So, integral is rather a function of the oriented curve defined by a path than of the path itself.)

3.1.11. The integral along a path is estimated by the corresponding arclength integral: if \( \gamma \) is a piecewise smooth path and \( f \) is a function continuous on \( |\gamma| \), then
\[
\left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f| \, dz \leq \sup_{|\gamma|} |f| \cdot \text{length}(\gamma).
\]

3.1.12. Integration along a path is “continuous” with respect to the uniform convergence of functions: if \( \gamma \) is a piecewise smooth path and a sequence \( (f_n) \) of continuous functions converges uniformly on \( |\gamma| \) to a function \( f \), then \( \int_{\gamma} f_n(z) \, dz \to \int_{\gamma} f(z) \, dz \).
3.2. Integration via primitive functions

3.2.1. Theorem. If a continuous function \( f \) on an open set \( U \subseteq \mathbb{C} \) has a primitive \( F \) in \( U \), then for any piecewise smooth path \( \gamma \) in \( U \) with the initial point \( z_1 \) and the terminal point \( z_2 \), \( \int_{\gamma} f(z) \, dz = F(z_2) - F(z_1) \).

3.2.2. Corollary. If a function \( f \) has a primitive in an open set \( U \), then for any piecewise smooth curve \( \gamma \) in \( U \), \( \int_{\gamma} f(z) \, dz \) only depends on the endpoints of \( \gamma \); equivalently, \( \int_{\gamma} f(z) \, dz = 0 \) for any piecewise smooth loop \( \gamma \) in \( U \).

3.2.3. Corollary. For any \( n \geq 0 \) and any piecewise smooth loop \( \gamma \) in \( \mathbb{C} \), and for any \( n \leq -2 \) and any piecewise smooth loop \( \gamma \) in \( \mathbb{C}^* \), \( \int_{\gamma} z^n \, dz = 0 \).

Notice that \( \int_{|z|=1} z^{-1} = 2\pi i \neq 0 \), so \( z^{-1} \) has no primitive in \( \mathbb{C}^* \).

3.2.4. Assume that a function \( f \) has a primitive \( F \) in an open set \( U \) and \( \gamma \) be a (continuous) path in \( U \) with the initial point \( z_1 \) and the terminal point \( z_2 \); we then define \( \int_{\gamma} f(z) \, dz = F(z_2) - F(z_1) \). If \( \gamma \) is a piecewise smooth path, this definition coincides with that in 3.1.6.

3.2.5. The converse is also true:

**Theorem.** Let \( f \) be a continuous function on a domain \( D \) such that \( \int_{\gamma} f(z) \, dz = 0 \) for any piecewise smooth loop \( \gamma \) in \( D \). Then \( f \) has a primitive in \( D \), defined by \( F(z) = \int_{\gamma} f(z) \, dz \) for any piecewise smooth curve \( \gamma \) in \( D \) that begins at a fixed point \( z_0 \) and ends at \( z \in D \).

Moreover, the same remains true if “piecewise smooth loops” are replaced by “polygonal loops”, or even by “rectangular loops” in \( D \):

**Theorem.** Let \( f \) be a continuous function on a domain \( D \) such that \( \int_{\gamma} f(z) \, dz = 0 \) for any rectangular closed path \( \gamma \) in \( D \). Then \( f \) has a primitive in \( D \).

3.3. Integration via local primitives

3.3.1. We say that a function \( f: U \subseteq \mathbb{C} \) on an open set \( U \subseteq \mathbb{C} \) has local primitives in \( U \) if every \( z \in U \) has a neighborhood \( V \subseteq U \) such that \( f \) has a primitive in \( V \).

3.3.2. Let \( f \) be a function on an open set \( U \) and let \( \gamma: [a, b] \to D \) be a continuous path in \( U \). A function \( \Phi: [a, b] \to \mathbb{C} \) is called a primitive of \( f \) along \( \gamma \) if for every \( t_0 \in [a, b] \) there is a primitive \( \Phi \) of \( f \) in a neighborhood of \( \gamma(t_0) \) such that \( \Phi(\gamma(t)) = \Phi(t) \) in a neighborhood of \( t_0 \).

3.3.3. Theorem. Let \( f \) be a function on an open set \( U \) and \( \gamma \) be a continuous path in \( U \). If \( f \) has local primitives in \( U \), then \( f \) has a primitive \( \Phi \) along \( \gamma \), and a function \( \Psi \) is a primitive of \( f \) along \( \gamma \) if \( \Psi = \Phi + \text{const} \).

**Proof.** Let \( \gamma: [a, b] \to U \). For any subinterval \( I \) of \([a, b]\), let’s say that \( \Phi \) is a primitive of \( f \) on \( I \) if \( \Phi \) is a primitive of \( f \) along \( \gamma \) on \( I \).

The set \( U \), and \( |\gamma| \) in particular, are covered by open sets on which \( f \) has a primitive; since \( |\gamma| \) is compact, there are finitely many of such sets, \( V_1, \ldots, V_k \), that cover \( |\gamma| \): \( |\gamma| \subseteq \bigcup_{i=1}^k V_k \). Then \( [a, b] \) is covered by the open subsets \( \gamma^{-1}(V_i), i = 1, \ldots, k \). Partition \([a, b]\) to small intervals, \( a = t_0 < t_1 < \ldots < t_m = b \), such that for every \( j \), \([t_{j-1}, t_j] \subseteq \gamma^{-1}(V_i) \) for some \( i \). Then \( f \) has a primitive on \([t_{j-1}, t_j]\) for every \( j \). Choose such primitives, \( \Phi_1, \ldots, \Phi_k \), and after adding, if necessary, certain constants to these functions, achieve that \( \Phi_j(t) = \Phi_{j+1}(t_j) \) for all \( j \). Define \( \Phi \) on \([a, b]\) such that \( \Phi |_{[t_{j-1}, t_j]} = \Phi_j \) for all \( j \); then \( \Phi \) is continuous. It only remains to check that \( \Phi \) is a primitive along \( \gamma \) at the points \( t_j \). Let \( c = t_j \) for some \( j \) and let \( F_1, F_2 \) be the primitives of \( f \) in some discs \( \Delta_1, \Delta_2 \) centered at \( \gamma(c) \) such that \( \Phi_j(t) = F_1(\gamma(t)) \) for \( t \in [t_{j-1}, c] \) and \( \Phi_{j+1}(t) = F_2(\gamma(t)) \) for \( t \in [c, t_{j+1}] \). Let \( \Delta \) be the smallest of \( \Delta_1, \Delta_2 \). Then \( F_1 - F_2 = \text{const on } \Delta \), and since \( F_1(\gamma(c)) = F_1(c) = F_2(c) = F_2(\gamma(c)) \) we have that \( F_1 = F_2 \) on \( \Delta \) and so, \( \Phi \) is a primitive of \( f \) on any interval \( I \cap \Delta \) for which \( |\gamma(I)| \subseteq \Delta \).

Let’s now show that the primitive of \( f \) on \([a, b]\) is unique, up to a constant. Suppose that \( \Phi_1 \) and \( \Phi_2 \) are two such primitives, with \( \Phi_1(a) = \Phi_2(a) \). Let \( c = \sup \{ t : \Phi_1(t) = \Phi_2(t) \} \) (this set is nonempty since \( \Phi_1(a) = \Phi_2(a) \)), and suppose that \( c < b \). Let \( F_1, F_2 \) be the primitives of \( f \) in some discs \( \Delta_1, \Delta_2 \) centered at \( \gamma(c) \) such that \( \Phi_1(t) = F_1(\gamma(t)) \) in a neighborhood of \( c \). Let \( \Delta \) be the smallest of \( \Delta_1, \Delta_2 \). Then
\[ F_1 - F_2 = \text{const on } \Delta, \text{ and since } F_1(\gamma(t)) = \Phi_1(t) = \Phi_2(t) = F_2(\gamma(t)) \text{ for the points } t < c \text{ in a neighborhood of } c (\text{or at } c \text{ itself if } c = a), \text{ we have that } F_1 = F_2 \text{ on } \Delta \text{ and so, } \Phi_1(t) = F_1(\gamma(t)) = F_2(\gamma(t)) = \Phi_2(t) \text{ for the points } t > c \text{ in a neighborhood of } c, \text{ which contradicts the choice of } c. \]

3.3.4. Let \( f \) be a function on an open set \( U, \gamma:[a,b] \rightarrow D \) be a continuous path in \( U \), and assume that \( f \) has a primitive \( \Phi \) along \( |\gamma| \). Then we define \( \int_{\gamma} f(z) \, dz = \Phi(b) - \Phi(a) \).

In the case \( \gamma \) is piecewise smooth and \( f \) has a primitive along \( \gamma \), this definition of \( \int_{\gamma} f(z) \, dz \) coincides with the introduced above.

3.3.5. The properties 3.1.7, 3.1.8, 3.1.9, 3.1.10, and 3.1.12 of \( f \) hold true. (The property 3.1.11 also holds, but the arc-length integral \( \int_{\gamma} |f(z)| \, |dz| \) may equal \( \infty \).)

3.3.6. Theorem. Let \( f \) be a function on an open set \( U \) that has local primitives in \( U \). Then \( \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz \) for any two paths \( \gamma_1 \) and \( \gamma_2 \) homotopic in \( U \) as paths with common endpoints, or as closed paths.

3.3.7. Corollary. Let \( f \) be a function on an open set \( U \) that has local primitives in \( U \). Then for any contractible path \( \gamma \) in \( U \), \( \oint_{\gamma} f(z) \, dz = 0 \).

3.3.8. Theorem. If \( f \) has local primitives in a simply connected domain \( D \), then \( f \) has a primitive in \( D \).

### 4. Cauchy’s theorems and their corollaries

4.1. The local and the global Cauchy’s theorems

4.1.1. Cauchy’s theorem says that if \( f \) is a holomorphic function on a domain \( D \) and \( G \subseteq D \) is a Jordan domain in \( D \) enclosed by a Jordan curve \( \gamma \), then \( \int_{\gamma} f(z) \, dz = 0 \). There is a simple proof of this theorem based on Green’s formula and the Cauchy-Riemann equations; however, this proof requires \( f \) to be continuously differentiable, and \( \gamma \) to be piecewise smooth. And anyway, a self-contained proof is preferable.

4.1.2. The local Cauchy theorem. Let \( f \) be a holomorphic function on a disc \( \Delta \). Then for any rectangle \( R \subset \Delta, \oint_{\partial R} f(z) \, dz = 0 \).

4.1.3. Corollary. If \( f \) is a holomorphic function on a disc \( \Delta \), then \( f \) has a primitive in \( \Delta \).

4.1.4. So, if \( f \) is a holomorphic function on a domain \( D \), then \( f \) has local primitives in \( D \), and so, \( \int_{\gamma} f(z) \, dz \) is defined for every continuous path in \( D \).

4.1.5. The global Cauchy theorem. Let \( f \) be a holomorphic function on a domain \( D \). Then for any two paths \( \gamma_1 \) and \( \gamma_2 \) homotopic in \( D \) (as paths with common endpoints or as closed paths), \( \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz \).

4.1.6. Theorem. If \( f \) is a holomorphic function on a domain \( D \), then for any contractible loop \( \gamma \) in \( D \), \( \oint_{\gamma} f(z) \, dz = 0 \). In particular, if \( D \) is simply connected, then for any loop \( \gamma \) in \( D \), \( \oint_{\gamma} f(z) \, dz = 0 \).

4.1.7. Theorem. If \( f \) is a holomorphic function on a simply connected domain \( D \), then \( f \) has a primitive in \( D \).

4.1.8. If \( f \) is a holomorphic function on a domain \( D \) and \( \sigma = \sum_{j=1}^{k} m_j \gamma_j \) is a cycle in \( D \), we define \( \oint_{\sigma} f(z) \, dz = \sum_{j=1}^{k} m_j \oint_{\gamma_j} f(z) \, dz \).

4.1.9. The global Cauchy theorem for cycles. Let \( f \) be a holomorphic function on a domain \( D \). Then for any two cycles \( \sigma_1 \) and \( \sigma_2 \) homologous in \( D \), \( \oint_{\sigma_1} f(z) \, dz = \oint_{\sigma_2} f(z) \, dz \).

4.2. Cauchy’s integral formula

4.2.1. Cauchy’s integral formula for simple loops. Let \( f \) be a holomorphic function on a domain \( D \) and let \( \gamma \) be a simple loop contractible in \( D \). Then for any \( z \in \Delta \) encircled by \( \gamma \), \( \oint_{\gamma} \frac{f(w)}{w-z} \, dw = 2\pi if(z) \).

4.2.2. There is a strengthening of this theorem where \( f \) is only assumed to be holomorphic in the domain \( D \) encircled by \( \gamma \) and continuous in \( \overline{D} \) (Goursat’s theorem). We can easily obtain the following its special case:
**Theorem.** Let \( \gamma \) be a piecewise smooth simple loop in \( \mathbb{C} \), such that the domain \( D \) encountered by \( \gamma \) is convex or is starlike with respect to a point \( z_0 \in D \), and let \( f \) be a function continuous in \( D = D \cup |\gamma| \) and holomorphic in \( D \). Then for any \( z \in D \), \( \oint_{\gamma} \frac{f(w)}{w-z} \, dw = 2\pi i f(z) \).

4.2.3. Cauchy’s integral formula for general loops. Let \( f \) be a function holomorphic in a domain \( D \) and let \( \gamma \) be a loop contractible in \( D \). Then for any \( z \in D \setminus |\gamma| \), \( \oint_{\gamma} \frac{f(w)}{w-z} \, dw = 2\pi i \text{Im}(\gamma) f(z) \).

4.2.4. Corollary. Let \( \gamma \) be a loop contractible in \( \mathbb{C} \). Then for any \( z \in \mathbb{C} \setminus |\gamma| \), \( n(\gamma, z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dw}{w-z} \).

4.3. Infinite differentiability of holomorphic functions and Morera’s theorem

4.3.1. Theorem. Let \( D \) be a domain, \( z_0 \in D \), \( \gamma \) be a piecewise smooth path in \( \mathbb{C} \), and \( K(z, w) \) be a function on \( D \times |\gamma| \) such that for every \( z \in D \), \( K(z, w) \) is a continuous function of \( w \in |\gamma| \), for every \( w \in |\gamma| \), \( K(z, w) \) is differentiable with respect to \( z \) at \( z_0 \), and the convergence \( \frac{K(z, w) - K(z_0, w)}{w - z_0} \to \frac{\partial K}{\partial z}(z_0, w) \) is uniform on \( w \in |\gamma| \). Then the function \( f(z) = \int_{\gamma} K(z, w) \, dw \) is differentiable at \( z_0 \) with \( f'(z_0) = \int_{\gamma} \frac{\partial K}{\partial z}(z_0, w) \, dw \).

4.3.2. As a special case we have:

**Theorem.** Let \( \gamma \) be a piecewise smooth path in \( \mathbb{C} \) and let \( g \) be a continuous function on \( |\gamma| \). Then the function \( f(z) = \int_{\gamma} g(w) \, dw \) is holomorphic in \( \mathbb{C} \setminus |\gamma| \) with \( f'(z) = \int_{\gamma} g(w) \, dw \).

4.3.3. Theorem. If \( f \) is a holomorphic function on an open set \( U \), then \( f' \) is also holomorphic in \( U \), and so are \( f^{(n)} \) for all \( n \). For each \( n \in \mathbb{N} \) and every \( z \in U \), \( f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{n+1}} \, dw \), where \( \gamma \) is any simple loop in \( D \) encircling \( z \).

4.3.4. Morera’s theorem. If a function \( f \) has a primitive, or local primitives, in an open set \( U \), then \( f \) is holomorphic in \( U \).

4.3.5. Morera’s theorem. Let \( f \) be a continuous function on an open set \( U \) such that \( \oint_{\partial R} f(z) \, dz = 0 \) for every rectangle \( R \subset U \). Then \( f \) is holomorphic in \( U \).

4.3.6. As a corollary we also get:

**Theorem.** If a function \( f \) is holomorphic in \( U \setminus \{z_0\} \) for some open set \( U \) and \( z_0 \in U \) and is bounded in a neighborhood of \( z_0 \), then \( f \) is differentiable at \( z_0 \) as well.

4.4. An estimate of the derivatives and Liouville’s theorem

4.4.1. Theorem. If a function \( f \) is holomorphic in a disc \( \Delta(z_0, r) \) and \( \sup_{\Delta(z_0, r)} |f| = M \), then for every \( n \in \mathbb{N} \), \( |f^{(n)}(z_0)| \leq \frac{nM}{r^n} \).

4.4.2. Liouville’s theorem. Any bounded entire function is constant.

4.4.3. The fundamental theorem of algebra. Every nonconstant polynomial over \( \mathbb{C} \) has a root in \( \mathbb{C} \).

4.4.4. Theorem. If an entire function \( f \) satisfies \( |f(z)| < M|z|^d + N \), \( z \in \mathbb{C} \), for some \( d \in \mathbb{N} \) and \( M, N > 0 \), then \( f \) is a polynomial of degree \( \leq d \).

4.5. The mean value theorem, the maximum principle, and Schwarz’s lemma

4.5.1. The mean value theorem. Let a function \( f \) be holomorphic in a disc \( \Delta = \Delta(z_0, r) \subset \mathbb{C} \) and continuous in its closure \( \Delta \). Then \( f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt \).

4.5.2. The maximum principle. Let \( f \) be a nonconstant holomorphic function on a domain \( D \). Then none of the functions \( |f|, \text{Re} f, \text{Im} f \) attains its maximum value in \( D \).

4.5.3. Corollary. Let \( f \) be a function holomorphic in a bounded domain \( D \) and continuous on \( \overline{D} \). Then the functions \( |f|, \text{Re} f, \text{Im} f \) attain their maximal values on \( \overline{D} \) at a point of \( \partial D \).

4.5.4. The minimum principle. Let \( f \) be a nonconstant holomorphic function on a domain \( D \) with \( f(z) \neq 0 \) for all \( z \in C \). Then the function \( |f| \) does not attain its minimum value in \( D \).
4.5.5. **Schwarz’s lemma.** Let $f$ be a holomorphic function on the unit disc $\Delta = \Delta(0,1)$, satisfying $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \Delta$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \Delta$. Moreover, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in \Delta$, then for some $c \in \mathbb{C}$ with $|c| = 1$, $f(z) = cz$ for all $z \in \Delta$.

4.6. **The functions $\log f$ and $\sqrt{f}$**

4.6.1. **Theorem.** Let $f$ be a holomorphic function on a domain $D$ and let $\gamma$ be a loop in $D$ such that $f(z) \neq 0$ for all $z \in |\gamma|$. Then $\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = n(f_\gamma, 0)$.

4.6.2. **Theorem.** Let $f$ be a holomorphic function on a domain $D$ such that $f(z) \neq 0$ for all $z \in D$. Then a branch of the function $\log f$ exists on $D$ iff $\frac{f'(z)}{f(z)} \neq 0$ for every loop $\gamma$ in $D$.

4.6.3. **Theorem.** Let $f(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$ where $c \in \mathbb{C}$ is nonzero, $z_1, \ldots, z_k \in \mathbb{C}$ are pairwise distinct, and $m_1, \ldots, m_k \in \mathbb{Z}$. Then a branch of the function $\log f$ exists on a domain $D \subseteq \mathbb{C} \setminus \{z_1, \ldots, z_k\}$ iff \(\sum_{i=1}^{k} m_i n(\gamma, z_i) = 0\) for every loop $\gamma$ in $D$.

4.6.4. **Theorem.** Let $f$ be a holomorphic function on a domain $D$ such that $f(z) \neq 0$ for all $z \in D$. Then a branch of the function $\sqrt{f}$ exists on $D$ iff $\frac{f'(z)}{2f(z)} \neq 0$ for every loop $\gamma$ in $D$.

4.6.5. **Theorem.** Let $f(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$ where $c \in \mathbb{C}$ is nonzero, $z_1, \ldots, z_k \in \mathbb{C}$ are pairwise distinct, and $m_1, \ldots, m_k \in \mathbb{Z}$. Then a branch of the function $\sqrt{f}$ exists on a domain $D \subseteq \mathbb{C} \setminus \{z_1, \ldots, z_k\}$ iff $\sum_{i=1}^{k} m_i n(\gamma, z_i) = 0$ for every loop $\gamma$ in $D$.

5. **Harmonic functions**

5.1. **Harmonic functions and holomorphic functions**

5.1.1. A real-valued function $u$ on an open set $U \subseteq \mathbb{C}$ is said to be harmonic if $u \in C^2(U)$ and satisfies the Laplace equation \(\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\).

5.1.2. **Theorem.** If $f$ is a holomorphic function on an open set $U$, then both $\text{Re} f$ and $\text{Im} f$ are harmonic functions on $U$.

5.1.3. Let $u$ be a harmonic function on a domain $D$. A real-valued function $v$ on $D$ is said to be a harmonic conjugate of $u$ if the function $u + iv$ is holomorphic in $U$. A harmonic conjugate, if exists, is unique up to a constant.

5.1.4. **Theorem.** If $u$ is a harmonic function in a domain $D$, then local conjugates of $u$ exist locally in $D$: for every point $z \in D$ a harmonic conjugate of $u$ exists in a neighborhood of $z$. If $D$ is a simply connected domain, then $u$ has a (global) harmonic conjugate in $D$.

5.1.5. **Corollary.** If $u$ is a harmonic function in an open set $U$, then $u$ is infinitely differentiable in $U$, $u \in C^\infty(U)$.

5.1.6. **Corollary.** The composition $u \circ g$ of a holomorphic function $g$ and a harmonic function $u$ is harmonic.

5.1.7. **The mean value theorem for harmonic functions.** If function $u$ is harmonic in a disc $\Delta(z_0, r)$ and continuous on $\partial \Delta(z_0, r)$, then $u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{it}) \, dt$.

5.1.8. **The maximum principle for harmonic functions.** If $u$ is a nonconstant harmonic function on a domain $D$, then $u$ does not attain its maximal value in $U$.

5.1.9. **Corollary.** If a function $u$ is harmonic in a bounded domain $D$ and is continuous in $\overline{D}$, then $u$ takes its maximal value on $\partial D$ at a point of $\partial D$.

5.1.10. **Remark.** The maximum principle is derived from the mean value theorem: any function with the mean value property satisfies the maximum principle.
5.2. Dirichlet’s boundary problem for harmonic functions in a disc

5.2.1. Let \( D \) be a domain in \( \mathbb{C} \) and let \( h \) be a continuous function on \( \partial D \). The Dirichlet boundary problem is the problem of finding a function \( u \) which is harmonic in \( D \), continuous on \( D \), and is such that \( u|_{\partial D} = h \). If a domain \( D \) is such that Dirichlet’s problem is solvable for every \( h \in C(\partial D) \), then \( D \) is said to be regular for Dirichlet’s problem.

5.2.2. Let \( \Delta = \Delta(0,1) \), and assume that a continuous function \( h \) on \( K = \partial \Delta \) is representable as the sum of its Fourier series, \( h(e^{i\theta}) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \). Then the function \( u \) on \( \Delta \) defined by

\[
u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi,
\]
solves Dirichlet’s problem in \( \Delta \) for \( h \).

5.2.3. The Poisson kernel for the unit disc is the function \( P(z,w) = \frac{1-|z|^2}{|w-z|^2} \), where \( z \in \Delta(0,1) \), \( w \in K(0,1) \). For \( z = |z|e^{i\theta} \) and \( w = e^{it} \), \( P \) can be written in the form \( P(z,w) = \frac{1-|z|^2}{1+|z|^2-2|z|\cos(t-\theta)} \).

5.2.4. The Poisson kernel equals \( \Re \frac{w+z}{w-z} \), so for every \( w \), \( P \) is harmonic with respect to \( z \). For every \( z \in \Delta(0,1) \), \( \frac{1}{2\pi} \int_{0}^{2\pi} P(z,e^{it}) \ dt = 1 \).

5.2.5. For a general disc \( \Delta(z_0,r) \), the Poisson kernel is the function \( P((z-z_0)/r, (w-z_0)/r) = \frac{r^2-|z-z_0|^2}{|w-z|^2} \), \( z \in \Delta(z_0,r) \), \( w \in K(z_0,r) \).

5.2.6. Schwarz’s theorem. Any disc is regular for Dirichlet’s problem: For any continuous function \( h \) on a circle \( K(z_0,r) \) the function \( u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} h(z_0 + re^{it})P((z-z_0)/r, e^{it}) \ dt \) is harmonic in \( \Delta(z_0,r) \) and for every \( w \in K(z_0,r) \), \( \lim_{z \to w} u(z) = h(w) \).

5.2.7. As a corollary of Schwarz’s theorem, we have:

Theorem. Every continuous real-valued function on an open set \( U \) that satisfies the mean value theorem in \( U \) is harmonic in \( U \).

5.2.8. Another corollary is

Theorem: Harnack’s inequalities. If a function \( u \) is harmonic and nonnegative in a disc \( \Delta(z_0,r) \), then for all \( z \in \Delta(z_0,r) \), \( \frac{r}{r+|z-z_0|} u(z) \leq u(z) \leq \frac{r+|z-z_0|}{r} u(z_0) \).

5.2.9. We will see later that the fact that Dirichlet’s problem is solvable in discs implies that it is solvable for a bunch of simply connected domains.

6. Sequences and series of holomorphic functions, analytic functions, Taylor and Laurent series

6.1. Sequences and series of holomorphic functions

6.1.1. Theorem. Let a sequence \( (f_n) \) of holomorphic functions on an open set \( U \) converge normally on \( U \) to a function \( f \). Then \( f \) is holomorphic in \( U \) and \( f_n^{(k)} \to f^{(k)} \) normally on \( U \) for every \( k \in \mathbb{N} \).

6.1.2. Theorem. If a series \( \sum f_n \) of holomorphic functions on an open set \( U \) converges normally on \( U \) and \( f = \sum_{n=1}^{\infty} f_n \), then \( f \) is holomorphic in \( U \), and for every \( k \in \mathbb{N} \), \( f^{(k)} = \sum_{n=1}^{\infty} f_n^{(k)} \) where the series \( \sum f_n^{(k)} \) converges normally on \( U \).

6.1.3. Montel’s theorem. A family of holomorphic functions on an open set \( U \) is normal iff it is locally bounded.

6.1.4. Corollary. If a sequence \( (f_n) \) of holomorphic functions on an open set \( U \) converges pointwise on \( U \) to a function \( f \) and is locally bounded on \( U \), then \( (f_n) \) converges to \( f \) normally on \( U \).
6.1.5. Example. The sequence \((1 + \frac{z}{n})^n\), \(n \in \mathbb{N}\), of polynomials converges normally in \(C\) to the function \(e^z\).

6.1.6. Theorem. If a series \(\sum f_n\) of holomorphic functions on an open set \(U\) converges absolutely pointwise on \(U\) and the function \(\sum_{n=1}^\infty |f_n|\) is locally bounded in \(U\), the the series \(\sum_{n=1}^\infty f_n\) converges normally on \(U\).

6.1.7. Example. Riemann’s Zeta function \(\zeta(z) = \sum_{n=1}^\infty n^{-z}\) is holomorphic in the domain \(\{z : \text{Re} z > 1\}\).

6.1.8. Theorem. If functions \(f_t\), \(t \in (a,b)\) with \(a,b \in \mathbb{R} \cup \{-\infty, \infty\}\), are holomorphic in an open set \(U\), for all \(z \in U\) the integral \(f(z) = \int_a^b f_t(z) dt\) exists, and the integrals \(\int_a^b |f_t(z)| dt\) are locally bounded in \(U\), then \(f\) is holomorphic in \(U\) and \(f^{(k)}(z) = \int_a^b f_t^{(k)}(z) dt\) for all \(z \in U\) and all \(k \in \mathbb{N}\).

6.1.9. Example. The Gamma function \(\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt\) is holomorphic in the right half-plane \(\{z : \text{Re} z > 0\}\).

6.2. Power series and analytic functions

6.2.1. A series of the form \(\sum_{n=0}^\infty a_n(z-z_0)^n\), with \(z_0 \in \mathbb{C}\) and \(a_0, a_1, \ldots \in \mathbb{C}\), is called a power series centered at \(z_0\). The number \(R = (\limsup_{n \to \infty} \sqrt[|a_n|]{n})^{-1} \in [0, \infty]\) is called the radius of convergence of the series, and, in the case \(R > 0\), the disc \(\Delta(z_0, R)\) is called the disc of convergence of the series.

6.2.2. Theorem. Assume that the radius of convergence of a power series \(\sum_{n=0}^\infty a_n(z-z_0)^n\) is positive and let \(\Delta = \Delta(z_0, R)\) be its disc of convergence. Then the series converges absolutely normally in \(\Delta\) and diverges at every point of \(\mathbb{C} \setminus \Delta\); the function \(f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n\) is holomorphic in \(\Delta\), and for every \(k \in \mathbb{N}\), \(f^{(k)}(z) = \sum_{n=k}^\infty \frac{n!}{(n-k)!}a_n(z-z_0)^{n-k}\) for all \(z \in \Delta\).

Notice that when \(R = \infty\), \(\Delta(z_0, R) = \mathbb{C}\).

6.2.3. The following theorem implies that if a power series converges at a point of the boundary of its disc of convergence, then the function defined by the series is “continuous” at this point if we don’t approach it to closely to the boundary:

Abel’s theorem. Let \(f(z) = \sum_{n=0}^\infty a_nz^n\) in a disc \(\Delta(z_0, R)\) and assume that the series converges at a point \(w \in K(z_0, R)\). Let \(M > 0\) and let \(L = \{z : |z-w|/(R-|z-z_0|) < M\}\) (Stolz’s region). Then
\[\lim_{z \to w, z \in L \cap \Delta(z_0, R)} f(z) = \sum_{n=0}^\infty a_nw^n.\]

6.2.4. A function \(f\) on an open set \(U\) is said to be analytic in \(U\) if for every \(z_0 \in U\) there exists \(R > 0\) such that \(f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n\) in \(\Delta(z_0, R)\) for some \(a_0, a_1, \ldots \in \mathbb{C}\).

If a function \(f\) is analytic in \(U\) then it is holomorphic in \(U\), and \(f’\) is also analytic in \(U\).

6.3. Holomorphic functions and Taylor series

6.3.1. Theorem. If a function \(f\) is holomorphic in a disc \(\Delta = \Delta(z_0, R)\), then \(f\) is representable in \(\Delta\) by a power series centered at \(z_0\), \(f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n\), where for every \(n\), \(a_n = \frac{1}{2\pi} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}}dw = \frac{1}{n!}f^{(n)}(z_0)\).

6.3.2. Let a function \(f\) be holomorphic in a neighborhood of a point \(z_0\). The power series \(\sum_{n=0}^\infty \frac{1}{n!}f^{(n)}(z-z_0)\) is called the Taylor series of \(f\) at \(z_0\).

6.3.3. Theorem. If a function \(f\) is holomorphic in an open set \(U\) then \(f\) is analytic in \(U\). For each \(z_0 \in U\), \(f\) is representable by its Taylor series in the disc \(\Delta(z_0, R)\), where \(R = \text{dist}(z_0, \partial U)\). Entire functions are representable by their Taylor series in entire \(\mathbb{C}\).

6.3.4. As a corollary, we have that the radius of convergence \(R\) of a power series \(f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n\) is the distance from \(z_0\) to the nearest point \(z_1 \in \mathbb{C}\) such that \(f\) has no holomorphic extension to a neighborhood of \(z_1\).

6.3.5. Suppose \(f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n\) and \(g(z) = \sum_{n=0}^\infty b_n(z-z_0)^n\) where the first series has radius convergence \(R_1\) and the second \(R_2\). Then
(i) \(f’(z) = \sum_{n=1}^\infty na_n(z-z_0)^{n-1}\), with radius of convergence \(R_1\).
We say that a continuous mapping $f$ is holomorphic if for every $z_0$ in its domain there exists a unique function $f(z)$ such that $f(z)$ is analytic in a neighborhood of $z_0$. The primitives of $f$ and $g$ are the functions $F(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$, with $c \in \mathbb{C}$, with radius of convergence $R_1$.

(ii) The primitives of $f$ are the functions $F(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$, with radius of convergence $\geq \min(R_1, R_2)$.

(iv) $(fg)(z) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) (z - z_0)^n$, with radius of convergence $\geq \min(R_1, R_2)$.

(v) If $a_0 = f(z_0) \neq 0$, then $(1/f)(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, where $c_0 = a_0^{-1}$ and for every $n \geq 1$, $c_n = -a_0^{-1} \sum_{k=0}^{n-1} a_k c_k$. The radius of convergence of this series is $\geq R_1$ if $f$ has no zeroes in $\Delta(z_0, R_1)$, and is equal to the distance from $z_0$ to the nearest zero of $f$ in this disc otherwise.

(vi) Suppose that $b(w) = \sum_{m=0}^{\infty} c_m (w - w_0)^m$ in $\Delta(w_0, R_3)$, where $w_0 = f(z_0) = a_0$. Then

$$(h \circ f)(z) = \sum_{m=0}^{\infty} c_m \left( \sum_{n=1}^{\infty} a_n (z - z_0)^n \right) = c_0 + c_1 a_1 (z - z_0) + (c_1 a_2 + c_2 a_1^2) (z - z_0)^2 + \cdots$$

(the series, obtained from the left hand side by opening parentheses), and the radius of convergence of this series is $\geq \min(R_1, d)$ where $d$ is such that $\sum_{n=1}^{\infty} |a_n| d^n = R_3$.

(vii) If $a_1 \neq 0$, then $f^{-1}(w) = \sum_{n=0}^{\infty} c_n (w - a_0)^n$ such that $c_0 = z_0$, $c_1 = a_1^{-1}$, $c_2 = -c_1 a_2/a_1^2$, ... in some neighborhood of $f(z_0) = a_0$.

6.3.6. (i) $(1 + z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n$ in $\Delta(0, 1)$.

(ii) $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ in $\mathbb{C}$.

(iii) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ in $\mathbb{C}$.

(iv) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ in $\mathbb{C}$.

(v) $\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n$ in $\Delta(0, 1)$.

(vi) $(1 + z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$ in $\Delta(0, 1)$.

(vii) $\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$ in $\Delta(0, 1)$.

(viii) $\arcsin z = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{1}{\binom{2n}{n}} z^{2n+1}$ in $\Delta(0, 1)$.

6.4. Zeroes and discreteness of holomorphic functions

6.4.1. If a function $f$ is holomorphic in a domain $D$ and is not identically zero in $D$, then for every $z_0 \in D$ there exists a unique $k \in \mathbb{N}$ such that $f$ is representable in the form $f(z) = b + (z - z_0)^k g(z)$ where $b = f(z_0)$ and $g(z_0) \neq 0$; this is so iff $f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$. In this case we say that $f$ takes the value $b$ with multiplicity $k$.

6.4.2. For a holomorphic function $f$, we say that a point $z_0$ is a zero of $f$ of order $k$ if $f$ takes the zero value at $z_0$ with multiplicity $k$.

6.4.3. We say that a continuous mapping $f: U \rightarrow Y$ is discrete if for every $y \in Y$ the set $f^{-1}(y)$ is discrete in $U$ (has no limit points in $U$).

Theorem. Every nonconstant holomorphic function is a discrete mapping.

6.4.4. It also follows that holomorphic functions possess the following “uniqueness property”:

Theorem. Let $f$ and $g$ be two holomorphic functions on a domain $D$ and assume that the set $\{ z : f(z) = g(z) \}$ has a limit point in $D$. Then $f = g$. 

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6.5. Laurent series

6.5.1. A two-sided power series \( \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \), with \( z_0 \in \mathbb{C} \) and \( a_0, a_1, \ldots \in \mathbb{C} \), is called a Laurent series centered at \( z_0 \). The numbers \( R = (\limsup_{n \to \infty} \sqrt[n]{|a_n|})^{-1} \in [0, \infty] \) and \( r = \limsup_{n \to -\infty} \sqrt[n]{|a_n|} \in [0, \infty] \) are called the outer and, respectively, the inner radius of convergence of the series.

6.5.2. Theorem. Let \( \sum_{n=0}^{\infty} a_n (z - z_0)^n \) be a Laurent series with outer and inner radii \( R \) and \( r \) and assume that \( R > r \). Then the series converges absolutely normally in the annulus \( A = \{ z : r < |z| < R \} \) and diverges at every point of \( \mathbb{C} \setminus \overline{A} \); the function \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \) is holomorphic in \( A \), and for every \( k \in \mathbb{N} \), \( f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1} \) for all \( z \in A \).

Notice that when the inner radius \( r = 0 \), the annulus \( A \) is the punctured disc \( \Delta^*(z_0, R) \); when the outer radius \( R = \infty \), \( A \) is the complement of a disc, \( A = \mathbb{C} \setminus \overline{\Delta(z_0, r)} \); and if both \( r = 0 \) and \( R = \infty \), then \( A = \mathbb{C}^* \).

6.5.3. Theorem. Let a function \( f \) be holomorphic in an annulus \( A = \{ z : r < |z| < R \} \) with \( 0 \leq r < R \leq \infty \). Then in \( A \), \( f \) is representable by the Laurent series, \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \), where for each \( n \), \( a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{n+1}} \, dw \) where \( \gamma \) is any simple loop in \( A \) encircling \( z_0 \).

7. Isolated singularities, residues and their applications

7.1. Classification of isolated singularities of holomorphic functions

7.1.1. If a function \( f \) is holomorphic in a punctured disc \( \Delta^* = \Delta^*(z_0, R) = \Delta(z_0, R) \setminus \{ z_0 \} \), then \( f \) is said to have an isolated singularity at \( z_0 \), and \( z_0 \) is said to be an isolated singular point of \( f \).

7.1.2. If \( f \) has an isolated singularity at \( z_0 \), then
- (i) if a finite \( \lim_{z \to z_0} f(z) \) exists, we say that \( f \) has a removable singularity at \( z_0 \);
- (ii) if \( \lim_{z \to z_0} f(z) = \infty \), we say that \( f \) has a pole at \( z_0 \);
- (iii) if \( \lim_{z \to z_0} f(z) \) does not exist, we say that \( f \) has an essential singularity at \( z_0 \).

7.1.3. Theorem. Let \( z_0 \) be an isolated singular point of a holomorphic function \( f \). Then
- (i) \( f \) has a removable singularity at \( z_0 \) iff \( f \) extends to a holomorphic function in a neighborhood of \( z_0 \) and the Laurent series of \( f \) at \( z_0 \) is actually a Taylor series, \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \);
- (ii) if the singularity of \( f \) at \( z_0 \) is not removable, then \( f \) has a pole at \( z_0 \) iff the Laurent series of \( f \) at \( z_0 \) has the form \( f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n \) for some \( k \in \mathbb{N} \), iff the function \( 1/f(z) \) has a removable singularity at \( z_0 \), and iff the function \( f(z)(z - z_0)^{-k} \) has a removable singularity at \( z_0 \) for some \( k \in \mathbb{N} \);
- (iii) \( f \) has an essential singularity at \( z_0 \) otherwise, that is, in the Laurent series \( \sum_{n=0}^{\infty} a_n (z - z_0)^n \) of \( f \) at \( z_0 \), \( a_n \neq 0 \) for infinitely many negative \( n \).

7.1.4. The Riemann extension theorem. Let \( z_0 \) be an isolated singular point of a holomorphic function \( f \). If \( f \) is bounded in a neighborhood of \( z_0 \), then the singularity of \( f \) at \( z_0 \) is removable.

7.1.5. Let \( f \) have a pole at \( z_0 \). The integer \( k \) such that the Laurent series of \( f \) at \( z_0 \) is \( \sum_{n=-k}^{\infty} a_n (z - z_0)^n \) with \( a_{-k} \neq 0 \) is called the order of the pole. \( f \) has pole of order \( k \) at \( z_0 \) iff \( 1/f(z) \) has a zero of order \( k \) at \( z_0 \) and iff the function \( f(z)(z - z_0)^k \) has a finite nonzero value at \( z_0 \).

A pole of order 1 is said to be a simple pole.

If \( \sum_{n=-k}^{\infty} a_n (z - z_0)^n \) is the Laurent series of \( f \) at \( z_0 \), then the sum \( \sum_{n=-k}^{-1} a_n (z - z_0)^n \) is called the principal part, or the singular part of \( f \) at \( z_0 \).

7.1.6. A function \( f \) is said to have only isolated singularities in an open set \( U \) if \( f \) is holomorphic in \( U \setminus S \), where \( S \) is a discrete subset of \( U \). A function \( f \) is said to be meromorphic in \( U \) if it only has poles in \( U \), that is, if \( f \) is holomorphic in \( U \setminus S \) where \( S \) is a discrete subset of \( U \) and has poles or removable singularities at the points of \( S \).
7.1.7. The set of meromorphic functions on a domain $D$ is denoted by $\mathcal{M}(D)$. If $f, g \in \mathcal{M}(D)$, then $f + g$, $fg$, and, if $g \neq 0$, $f/g$ also also meromorphic, that is, $\mathcal{M}(D)$ is a field.

7.1.8. The Casorati-Weierstrass theorem. If $f$ has an essential singularity at $z_0$, then for any neighborhood $V$ of $z_0$ the set $f(V)$ is dense in $\mathbb{C}$.

There is a much stronger result, the so-called Great Picard’s theorem, saying that if $f$ has an essential singularity at $z_0$, then in any punctured neighborhood of $z_0$, $f$ takes all complex values except, maybe, one.

7.2. Holomorphic functions on and to the Riemann sphere

7.2.1. The infinity $\infty$ is a point of $\hat{\mathbb{C}}$. There is a homeomorphism between the neighborhood $\hat{\mathbb{C}}^* = \hat{\mathbb{C}} \setminus \{0\}$ of $\infty$ and the complex plain $\mathbb{C}$, given by $z \mapsto z^{-1}$, $\infty \mapsto 0$. This homeomorphisms allows us to consider $w = z^{-1}$ as a “coordinate” near $\infty$, and extend the notions and theorems dealing with “finite” points of $\hat{\mathbb{C}}$ to the point $\infty \in \hat{\mathbb{C}}$ as well. In particular, a disc centered at $\infty$ is defined as a set of the form $\{w : |w| < r\} = \{z : |z| > r^{-1}\} \cup \{\infty\}$ for some $r > 0$, and such a “disc” plays the role of a neighborhood of $\infty$.

7.2.2. A function $f$ is continuous, or differentiable at $\infty$ if the function $f(w^{-1})$ is continuous or, respectively, differentiable at 0. A power series centered at $\infty$ is a power series of the form $\sum_{n=0}^{\infty} a_n w^n = \sum_{n=0}^{\infty} a_n z^{-n}$; a Laurent series is $\sum_{n=-\infty}^{\infty} a_n w^n = \sum_{n=-\infty}^{\infty} a_n z^{-n} = \sum_{n=-\infty}^{\infty} a_n z^{n}$. A function $f$ has a removable singularity, a pole, or an essential singularity at $\infty$ if the function $f(w^{-1})$ has the corresponding singularity at 0.

7.2.3. A function $f : U \rightarrow \hat{\mathbb{C}}$ is said to be continuous (respectively, differentiable) at a point $z_0 \in U$ with $f(z_0) = \infty$ if the function $1/f(z)$ is continuous (respectively, differentiable) at $z_0$. Thus, meromorphic functions on $U$ are just the holomorphic functions $U \rightarrow \hat{\mathbb{C}}$.

7.2.4. Liouville’s theorem says that $\mathcal{O}(\hat{\mathbb{C}}) = \mathbb{C}$, that is, that there are no nonconstant holomorphic functions on $\hat{\mathbb{C}}$. It is easy to see that the only functions holomorphic in $\mathbb{C}$ and having a pole at $\infty$ are polynomials, and that the only functions meromorphic on $\hat{\mathbb{C}}$ are rational functions, $\mathcal{M}(\hat{\mathbb{C}}) = \mathbb{C}(z)$.

7.3. Residues

7.3.1. Let $z_0$ be an isolated singular point of a function $f$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Laurent series of $f$ at $z_0$. The residue $\text{Res}_{z_0} f$ of $f$ at $z_0$ is defined to be the coefficient $a_{-1}$ of the series.

7.3.2. If $f$ has a simple pole at $z_0$, then $\text{Res}_{z_0} f = \lim_{z \to z_0} (z - z_0)^{-1} f(z)$. If $f$ has a pole of order $k$ at $z_0$, then $\text{Res}_{z_0} f = \frac{1}{(k-1)!} \lim_{z \to z_0} ((z - z_0)^k f(z))^{(k-1)}$.

7.3.3. The residue theorem. If a function $f$ is holomorphic with only isolated singularities in a domain $D$ (that is, $f$ is holomorphic in $D \setminus S$ where $S$ is a discrete subset of $D$) and $\gamma$ is a contractible loop in $D$ that avoids $S$, then

$$\oint_{\gamma} f(z) \, dz = 2\pi i \sum_{z \in S} n(\gamma, z) \text{Res}_{z} f.$$ 

The residue theorem is a great tool for calculating contour integrals of holomorphic functions, as well as conventional “real”, proper and improper, integrals of elementary functions.

7.3.4. Let $\infty$ be an isolated singular point of a function $f$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=\infty}^{\infty} a_n z^{-n}$ be the Laurent series of $f$ at $\infty$. The residue of $f$ at $\infty$, $\text{Res}_{\infty} f$, is defined to be $-a_{-1}$ (and not $a_{1}$, as one could guess. Here is the reason: the “residues” are not, actually, residues of a function $f$, but of a differential form $f(z) \, dz$; if we rewrite the differential form $f(z) \, dz$ in terms of $z^{-1}$, we get $-\sum_{n=\infty}^{\infty} a_n z^{-n} z^2 d(z^{-1}) = -\sum_{n=\infty}^{\infty} a_n z^{-n} z^2 d(z^{-1})$, and the coefficient of $(z^{-1})^{-1} = z^1$ in this series is $-a_{-1}$.) Under this definition, $\int_{|z|=R} f(z) = -2\pi i \text{Res}_{\infty} f$ for $R$ large enough.
7.3.5. **Theorem.** If a function \( f \) has only isolated singularities in \( \hat{\mathbb{C}} \) then the sum of the residues of \( f \) at all its singular points and at \( \infty \) is equal to zero.

7.4. **The argument and the branched covering principles; the open mapping, Rouché’s, and Hurwitz’s theorems**

In the theorems below, the assumption “Let \( f \) be a function holomorphic (or meromorphic) in a domain \( D \) and \( \gamma \) be a simple loop contractible in \( D' \)” can be replaced by the assumption “Let \( D \) be a Jordan domain bounded by a Jordan curve \( \gamma \), and let \( f \) be holomorphic (or meromorphic) in \( D \) and continuous in \( \overline{D} \),” but to make this change one needs Goursat’s theorem.

7.4.1. Let \( f \) be a nonzero meromorphic function on a domain set \( D \). The meromorphic function \( L(f) = f'/f \) is called the logarithmic derivative of \( f \). (Indeed, \( L(f) = (\log f)' \).) It has the property \( L(fg) = L(f) + L(g) \) and \( L(f/g) = L(f) - L(g) \) for two nonzero functions \( f \) and \( g \), and \( L(f^n) = nL(f) \) for a meromorphic function \( f \) and \( n \in \mathbb{Z} \).

We have \( L(z^n) = n/z, L(1 + z/a) = 1/(z + a), L(e^{h(z)}) = h'(z) \) for any meromorphic \( h \).

7.4.2. The function \( L(f) = f'/f \) has only simple poles, which are located at the points at which \( f \) has a zero or a pole. If \( f \) has a zero of order \( n \) at \( z_0 \), then \( \text{Res}_{z_0}(f'/f) = n \), if \( f \) has a pole of order \( n \) at \( z_0 \), then \( \text{Res}_{z_0}(f'/f) = -n \).

7.4.3. **The argument principle.** Let \( f \) be a function meromorphic in a domain \( D \) and \( \gamma \) be a simple loop contractible in \( D \) and containing neither zeroes nor poles of \( f \). Then \( n(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f}(z) \, dz = Z - P \), where \( Z \) is the number of zeroes of \( f \) inside \( \gamma \) (counting with orders) and \( P \) is the number of poles of \( f \) inside \( \gamma \) (counting with orders).

7.4.4. Applying the argument principle to the function \( f - w, w \in \mathbb{C} \), we get:

**Theorem.** Let \( f \) be a function meromorphic in a domain \( D \), let \( \gamma \) be a simple loop contractible in \( D \) and containing no poles of \( f \), and let \( P \) be the number of poles of \( f \) inside \( \gamma \) (counting with orders). Then for any \( w \in \mathbb{C} \setminus f(\{\gamma\}) \), \( f \) takes the value \( w \) inside \( \gamma \) at \( n(f(\gamma), w) + P \) points (counting with multiplicities).

7.4.5. **Rouché’s theorem.** Let \( f \) and \( h \) be functions holomorphic in a domain \( D \), let \( \gamma \) be a simple loop contractible in \( D \), and assume that \( |f| > |h| \) on \( \gamma \). Then the number of zeroes of the function \( f + h \) inside \( \gamma \) (counting with orders) is equal to the number of zeroes of \( f \) inside \( \gamma \) (counting with orders).

7.4.6. The following fact is a stronger version of Rouché’s theorem:

**Theorem.** Let \( f \) and \( g \) be functions holomorphic in a domain \( D \), let \( \gamma \) be a simple loop contractible in \( D \), and assume that \( |f - g| < |f| + |g| \) on \( \gamma \). Then \( f \) and \( g \) have an equal number of zeroes inside \( \gamma \) (counting with orders).

7.4.7. **The branched covering principle.** Let a function \( f \) be holomorphic in an open set \( U \) and take a value \( w_0 \) at a point \( z_0 \in U \) with multiplicity \( n \). Then there is a domain \( D \) containing \( z_0 \) and \( s > 0 \) such that \( f|_{D\setminus\{z_0\}} \) is an \( n \)-fold covering of \( \Delta^s(w_0, s) \): it is an \( n \)-to-1 mapping \( D \setminus \{z_0\} \to \Delta(w_0, s) \setminus \{w_0\} \), which is a local isomorphism (an invertible holomorphic mapping whose inverse is also holomorphic) at all points of \( D \setminus \{z_0\} \).

7.4.8. **The open mapping theorem.** Any holomorphic function is an open mapping: If a function \( f \) is holomorphic in an open set \( U \), then \( f(U) \) is open.

7.4.9. An injective holomorphic function is called univalent. If \( f \) is univalent in an open set \( U \), then \( f'(z) \neq 0 \) for all \( z \in U \): conversely, if \( f \) is holomorphic in \( U \) and \( f'(z) \neq 0 \) in \( U \), then \( f \) is locally univalent in \( U \).

7.4.10. **The inverse mapping theorem.** If \( f \) is a holomorphic univalent function on an open set \( U \), then the inverse function \( f^{-1} : f(U) \to U \) is also holomorphic.
7.4.11. Hurwitz’s theorem. Let \((f_n)\) be a sequence of functions holomorphic in an open set \(U\) that converges to a function \(f\) normally in \(U\). Assume that \(f \not\equiv 0\) and \(f\) has a zero in \(U\); then there exists \(N \in \mathbb{N}\) such that for all \(n > N\), \(f_n\) has a zero in \(U\).

7.4.12. Theorem. If a sequence \((f_n)\) of univalent holomorphic functions converges normally in an open set \(U\) to a function \(f\), then either \(f\) is constant, or is univalent.

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### 8. Geometric theorems

#### 8.1. Conformal isomorphisms and the Riemann mapping theorem

8.1.1. Recall (see subsection 2.4 above) that a differentiable mapping \(f: U \to V\), where \(U, V\) are open subsets of \(\mathbb{C}\), is said to be conformal if \(f\) preserves the angles between vectors; this is so iff \(f\) is holomorphic in \(U\) with \(f'\) not vanishing in \(U\). \(f\) is said to be a conformal isomorphism if \(f\) is conformal, invertible, and \(f^{-1}\) is conformal as well; this is so iff \(f\) is holomorphic, surjective and univalent. Two open sets \(U\) and \(V\) in \(\mathbb{C}\) are said to be conformally isomorphic if there is a conformal isomorphism \(U \to V\).

8.1.2. A conformal automorphism of an open set \(U \subseteq \mathbb{C}\) is a conformal isomorphism \(U \to U\). Conformal automorphisms of any open subset of \(\mathbb{C}\) form a group under the operation of composition.

Theorem. The conformal automorphisms of the complex plane \(\mathbb{C}\), the Riemann sphere \(\hat{\mathbb{C}}\), and of the unit disc \(\Delta(0,1)\) are the corresponding Möbius transformations of these domains.

8.1.3. The Riemann mapping theorem. Let \(D \subset \mathbb{C}\) be a simply connected domain, not equal to \(\mathbb{C}\). Then \(D\) is conformally isomorphic to the unit disc \(\Delta(0,1)\). For any \(z_0 \in D\) there exists a unique conformal isomorphism \(f: D \to \Delta(0,1)\) with \(f(z_0) = 0\) and \(f'(z_0) > 0\).

8.1.4. Corollary. Any two simply connected domains \(D_1\) and \(D_2\) in \(\hat{\mathbb{C}}\) distinct from \(\mathbb{C}\) and \(\hat{\mathbb{C}}\) are conformally isomorphic. Given \(z_1 \in D_1\), \(z_2 \in D_2\), and \(\theta \in (-\pi, \pi]\), there exists a unique conformal isomorphism \(f: D_1 \to D_2\) with \(f(z_1) = z_2\) and \(\arg f'(z_1) = \theta\).

#### 8.2. Boundary correspondence and the Carathéodory-Osgood theorem

8.2.1. The Carathéodory-Osgood theorem. If \(D\) is a Jordan domain in \(\mathbb{C}\) and \(\Delta = \Delta(0,1)\), then any conformal isomorphism \(D \to \Delta\) is extendible to a homeomorphism \(\overline{D} \to \overline{\Delta}\).

8.2.2. Corollary. Any Jordan domain in \(\mathbb{C}\) is regular for Dirichlet’s boundary problem. (See 5.2.1.)

8.2.3. Corollary. For any Jordan domains \(D_1\) and \(D_2\) in \(\mathbb{C}\), any three points \(z_1, z_2, z_3 \in \partial D_1\) listed counterclockwisely, and any three points \(w_1, w_2, w_3 \in \partial D_2\) listed counterclockwisely, there exists a unique homeomorphism \(f: \overline{D_1} \to \overline{D_2}\) such that \(f|_{D_1}\) is a conformal isomorphism \(D_1 \to D_2\) and \(f(z_i) = w_i\), \(i = 1, 2, 3\).

8.2.4. The following theorem is a sort of converse of The Carathéodory-Osgood theorem 8.2.1:

**The boundary correspondence principle.** Let \(D\) be a Jordan domain, let \(\gamma = \partial D\), and let a function \(f\) be holomorphic in \(D\) and continuous and injective on \(|\gamma|\). Then \(f\) is a conformal isomorphism between \(D\) and the interior of \(f(|\gamma|)\).

#### 8.3. The Schwarz-Cristoffel formula

8.3.1. The following theorem establishes a formula for a conformal isomorphism between the upper halfplane and an arbitrary polygon in \(\mathbb{C}\).
The Schwarz-Cristoffel formula. Let \( a_1, \ldots, a_n \in \mathbb{R} \cup \{ \infty \} \) with \( a_1 < \cdots < a_n \), let \( 0 < a_1, \ldots, a_n < 2 \) with \( \sum_{k=1}^{n} \alpha_k = 2 \), let \( A \in \mathbb{C}^* \), let \( f(z) = A \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} \), (or \( f(z) = A \prod_{k=1}^{n} (z - a_k)^{-\alpha_k - 1} \), if \( a_n = \infty \)), and let \( F \) be a primitive of (a branch of) \( f \) in the upper half-plane \( H = \{ z : \operatorname{Im} z > 0 \} \). Then \( F \) is a conformal isomorphism between \( H \) and the interior of an \( n \)-gon in \( \mathbb{C} \), and a homeomorphism between \( \overline{H} \cup \{ \infty \} \) and \( \overline{P} \); \( P \) has vertices at the points \( F(a_1), \ldots, F(a_n) \), listed counterclockwise, and angles \( \alpha_1 \pi, \ldots, \alpha_n \pi \) at these points. Conversely, any conformal isomorphism between \( H \) and the interior of an \( n \)-gon has this form.

8.4. The reflection principle

8.4.1. The Schwarz reflection principle. Let \( K \) be a circle (that is, a circle or a straight line) in \( \mathbb{C} \), let \( \rho \) be the reflection with respect to \( K \), let \( D \) be a domain symmetric with respect to \( K \) (that is, with \( \rho(D) = D \)), let \( I = D \cap K \), let \( D_1 \) and \( D_2 \) be the connected components of \( D \setminus K \). Let \( f \) be a continuous function \( D_1 \cup I \rightarrow \mathbb{C} \) which is holomorphic (or meromorphic) in \( D_1 \) and maps \( I \) to a circle \( K \) in \( \mathbb{C} \). Let \( \tilde{\rho} \) be the reflection with respect to \( K \); then \( f \) can be extended to a function holomorphic (or meromorphic) in \( D \) by putting \( f|_{D_2} = \tilde{\rho} f \circ \rho \).

8.4.2. The elliptic sine \( \text{sn} \) (with parameter \( k \)) is constructed in the following way: it is first defined on the rectangle \( P = [-a, a] \times [0, b] \) from 8.3.2 as the inverse of the function \( F \) from 8.3.2, and then is extended to the whole plane \( \mathbb{C} \) with the help of the reflection principle. \( \text{sn} \) is a meromorphic function, which has poles at the points \( 2na + (2m + 1)b, n, m \in \mathbb{Z} \). \( \text{sn} \) is odd and bi-periodic, with periods \( 4a \) and \( 2bi \): \( \text{sn}(z) = -\text{sn} \, z \), \( \text{sn}(z + 4a) = \text{sn} \, z \), \( \text{sn}(z + 2bi) = \text{sn} \, z \). (Moreover, \( \text{sn}(-x + iy) = -\overline{\text{sn}(x + iy)} \) and \( \text{sn}(x - iy) = \overline{\text{sn}(x + iy)} \) for all \( x, y \in \mathbb{R} \).)

8.4.3. The modular \( \lambda \) function is constructed in the following way. Let \( T \) be the “infinite triangle” in the Poincare disc \( \Delta(0, 1) \) with vertices at the points \( z_1 = 1, z_2 = e^{2\pi i/3}, z_3 = e^{4\pi i/3} \). (The sides of \( T \) are the arcs of circles connecting \( z_1, z_2, z_3 \) and orthogonal to \( K(0,1) \) at these points.) There exists a unique conformal isomorphism \( \lambda \) between \( T \) and the upper half-plane \( H \) that maps \( z_1, z_2, z_3 \) to the points \( 0, 1, \) and \( \infty \) respectively. Then, using the reflection principle, \( \lambda \) can be extended to a holomorphic function from \( \Delta(0,1) \) onto \( \mathbb{C} \setminus \{0,1\} \).

The sets of points where \( \lambda \) takes values \( 0, 1, \) and \( \infty \) are dense in \( K(0,1) \), thus \( \lambda \) is not extendible to a continuous function in any domain larger than \( \Delta(0,1) \).

8.4.4. The inverse of the modular function can be used to prove:

The little Picard theorem. Any nonconstant entire function misses at most one value in \( \mathbb{C} \).
Proof. Assume that an entire function $f$ does not take two values; composing $f$ with a linear function, we may assume that these values are 0 and 1. Then the multivalued function $\lambda^{-1}of$ is defined on $\mathbb{C}$; since $\mathbb{C}$ is simply connected, by the monodromy theorem (see below?) this function has a branch $g$, which is a holomorphic function $\mathbb{C} \rightarrow \Delta(0,1)$. By Liouville’s theorem, $g = \text{const}$, and so, $f = \text{const}$. ■

9. Holomorphic and meromorphic functions as infinite sums and products

9.1. Series of meromorphic functions

9.1.1. A series $\sum f_n$ of meromorphic functions on an open set $U \subseteq \mathbb{C}$ is said to converge normally in $U$ if for every compact subset $K \subseteq U$ there exists $N \in \mathbb{N}$ such that $f_n$ have no poles in $K$ for $n \geq N$ and the series $\sum_{n=0}^{\infty} f_n$ converges normally in $U$. (In this definition, “compact sets” can be replaced by “suitable neighborhoods of all points” of $U$.)

9.1.2. Theorem. Assume that a series $\sum f_n$ of functions meromorphic in an open set $U$ converges normally in $U$, and let $f = \sum_{n=0}^{\infty} f_n$. Then $f$ is meromorphic in $U$, and for every $k \in \mathbb{N}$, $f^{(k)} = \sum_{n=0}^{\infty} f_n^{(k)}$ also normally in $U$.

9.2. The Mittag-Leffler theorem

9.2.1. The following theorem says that for any discrete set $P$ in an open subset of $\mathbb{C}$ there is a meromorphic function that has poles, with prescribed singular parts, at the points of $P$ only:

The Mittag-Leffler theorem. Let $U \subseteq \mathbb{C}$ be an open set and let $(z_n)$ be a discrete (that is, with no limit points in $U$) sequence in $U$. For each $n$ let $S_n$ be a rational function of the form $S_n(z) = p_n(z)(z - z_n)^{-d_n}$ where $p_n$ is a polynomial of degree $< d_n$. Then there are rational functions $R_n$ with poles in $\mathbb{C} \setminus U$ such that the series $\sum_{n=1}^{\infty} (S_n - R_n)$ converges normally in $U$. The sum $f$ of the series is a meromorphic function on $U$ with poles at the points $z_n$ only, and for each $n$, the singular value of $f$ at $z_n$ is $S_n$. Any other function $f$ on $U$ with this property differs from $f$ by a function holomorphic in $U$.

In this theorem, the rational functions $R_n$ are chosen differently in two basic subcases:

(i) If $z_n \rightarrow \infty$ (which is always the case if $U = \mathbb{C}$), then $R_n$ are polynomials of degree $\leq d_n$, $n \in \mathbb{N}$.

(ii) If $(z_n)$ is a sequence in $U$ with $\delta_n = \text{dist}(z_n, \partial U) \rightarrow 0$, then $R_n$ have the form $R_n(z) = q_n(z)/(z - w_n)^{-d_n}$, where $w_n \in \partial U$ is such that $|w_n - z_n| = \delta_n$ and $q_n$ is a polynomial of degree $< d_n$, $n \in \mathbb{N}$.

9.2.2. Examples.

(i) \[ \frac{x^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}. \]

(ii) $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}, n \neq 0} \left( \frac{1}{z-n} + \frac{1}{z+n} \right)$.

9.2.3. Given $\omega_1, \omega_2 \in \mathbb{C}^*$ with $\omega_1/\omega_2 \notin \mathbb{R}$, the set $\Omega = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} = \{ n_1 \omega_1 + n_2 \omega_2, n_1, n_2 \in \mathbb{Z} \}$ is called a lattice in $\mathbb{C}$. The series \[ \sum_{\omega \in \Omega \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \]
converges absolutely normally in $\mathbb{C}$; the sum $\wp(z)$ of this series (which does not depend on the order of summation) is called the Weierstrass function corresponding to $\Omega$. $\wp$ is a meromorphic functions with poles of order 2 at the points of $\Omega$. The derivative of $\wp$ is $\wp'(z) = -\sum_{\omega \in \Omega \setminus \{0\}} \frac{2}{(z-w)^3}$ $\wp$ is an even function: $\wp(-z) = \wp(z)$. $\wp$ is a bi-periodic, or an elliptic function, with periods $\omega_1$ and $\omega_2$: $\wp(z + \omega_1) = \wp(z)$ and $\wp(z + \omega_2) = \wp(z)$.

9.3. Infinite products of holomorphic functions

9.3.1. Theorem. Let $D \subseteq \mathbb{C}$ be a domain, let $(f_n)$ be a sequence of holomorphic functions on $D$, and assume that the series $\sum \text{Log} f_n$ converges normally in $D$. Then the function $f = \prod_{n=1}^{\infty} f_n$ is holomorphic in $D$, and if $f \neq 0$, $f'/f = \sum (f'_n/f_n)$, which the series (of meromorphic functions) converges normally on $D$. 

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9.4. The Weierstrass factorization theorem

9.4.1. The following theorem allows to construct a holomorphic function by its zeroes:

The Weierstrass factorization theorem. Let $U \subseteq \mathbb{C}$ be an open set, let $(z_n)$ be a discrete sequence in $U$, and let $(m_n)$ be a sequence of positive integers. Then there exists a holomorphic function $f$ on $U$ such that for each $n$, $f$ has zero of order $m_n$ at $z_n$, and has no other zeroes in $U$. If $f$ is another such function, then $f/f$ is a zero-free holomorphic function on $U$.

In this theorem, the function $f$ is constructed as an infinite product, defined differently in two basic subcases:

(i) If $z_n \to \infty$ (which is always the case when $U = \mathbb{C}$) and $z_n \neq 0$ for all $n$, let $(d_n)$ is a sequence of positive integers such that $\sum m_n \left( \frac{r}{|z_n|} \right)^{d_n+1} < \infty$ for all $r$; then $f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n} \right)^{m_n} e^{m_n R_n(z)}$, where $R_n(z) = \frac{z}{z_n} + \frac{1}{2} \left( \frac{z}{z_n} \right)^2 + \cdots + \frac{1}{d_n} \left( \frac{z}{z_n} \right)^{d_n}$, $n \in \mathbb{N}$.

(ii) If $(z_n)$ is a sequence in $U$ with $d_n = \text{dist}(z_n, \partial U) \to 0$, let $w_n \in \partial U$ be such that $|w_n - z_n| = \delta_n$; then $f(z) = \prod_{n=1}^{\infty} \left( \frac{z-w_n}{z-w_n} \right)^{d_n} e^{m_n R_n(z)}$, where $R_n(z) = \left( \frac{z-w_n}{z-w_n} \right) + \frac{1}{2} \left( \frac{z-w_n}{z-w_n} \right)^2 + \cdots + \frac{1}{d_n} \left( \frac{z-w_n}{z-w_n} \right)^{d_n}$, and $d_n$ are chosen so that $\sum m_n \left( \frac{d_n}{z-w_n} \right)^{d_n+1} < \infty$ for all $z \in U$.

9.4.2. Corollary. Any entire function $f$ can be written in the form $f(z) = z^m e^{h(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n} \right)^{m_n} e^{R_n(z)}$ where $z_n$ are the zeroes of $f$, $R_n$ are polynomials, and $h$ is an entire function.

9.4.3. Example. $\sin(z) = \pi z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left( 1 - \frac{z}{n} \right)^{\pi/n}$. So, $\sin(z) = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right)$. 

9.4.4. Corollary. Let $U \subseteq \mathbb{C}$ be an open set. Any meromorphic function on $U$ is representable as a quotient $f/g$ where $f$ and $g$ are holomorphic functions on $U$.

9.5. The gamma and zeta functions

9.5.1. Euler's gamma function $\Gamma$ is defined by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ in the halfplane $\{ z : \Re z > 0 \}$, and is a holomorphic function there. $\Gamma$ satisfies $\Gamma(z+1) = z \Gamma(z)$, which implies that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. With the help of the identity $\Gamma(z) = \Gamma(z+1)/z$, $\Gamma$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at $0, -1, -2, \ldots$ only, and with $\text{Res}_0 \Gamma = 1$ and $\text{Res}_{-n} \Gamma = (-1)^n n!$ at these poles.

9.5.2. The Mittag-Leffler series for $\Gamma$ is $\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^\infty t^{z-1} e^{-t} dt$.

9.5.3. Euler's formula for $\Gamma$ is $\Gamma(z) = \lim_{n \to \infty} \frac{n^z n!}{(z+1)(z+2)\cdots(z+n)}$ (which works for all $z \in \mathbb{C}$).

9.5.4. $\Gamma$ has no zeroes. The Weierstrass product for $\Gamma$ is $\Gamma(z) = z^{\gamma-1} e^{-z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-\gamma} e^{z/n}$ where $\gamma$ is Euler's constant, $\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right)$. It follows that $\Gamma'(z)/\Gamma(z) = -\gamma - z - \sum_{n=1}^{\infty} \left( \frac{1}{z+n} - \frac{1}{n} \right)$.

9.5.5. For any $z$, $\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. In particular, $\Gamma(1/2) = \sqrt{\pi}$.

9.5.6. Riemann's zeta function $\zeta$ is defined by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, $\Re z > 1$; it is also equal to $\prod_{p \text{ prime}} \frac{1}{1 - p^{-z}}$.

9.5.7. $\zeta$ and $\Gamma$ satisfy $\zeta(z) = g(z)/\Gamma(z)$, where $g(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t-1} dt$. We have $g = g_1 + g_2$ where $g_1(z) = \int_1^\infty \frac{t^{z-1}}{e^t-1} dt$ and $g_2(z) = \int_0^1 \frac{t^{z-1}}{e^t-1} dt$. The function $g_1$ is entire. The function $g_2$ has the Mittag-Leffler representation $g_2(z) = \frac{1}{z-1} - \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{a_n}{z+n}$, where $|a_n| < M/n^s$ for some $M$ and all $n$, and $a_n = 0$ for all even $n$. Thus, $g_2$ is meromorphic in $\mathbb{C}$ with simple poles at $1, 0, -1, -3, -5, \ldots$. Since $1/\Gamma$ has simple zeroes at $0, -1, -2, \ldots$, we obtain that $\zeta$ is (extendible to) a meromorphic function in $\mathbb{C}$, with a single simple pole at $1$, and with zeroes at $-2, -4, \ldots$. The location of other zeroes of $\zeta$ is the subject of a central problem of mathematics, the so-called Riemann hypothesis.