Radical Pi presents:

Products of Sums of Squares

by Professor Daniel Shapiro

Suppose K is a field. For concreteness, think of K as the field of rational functions (quotients of polynomials) with real coefficients. For positive integer n, let $D_{K(n)}$ be the set of nonzero elements of K that can be expressed as a sum of n squares in K.

The sets $D_{K(1)}$ and $D_{K(2)}$ are closed under multiplication because of the formulas: $a^2b^2 = (ab)^2$ and $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$. A 4-square identity (due originally to Euler) shows that $D_{K(4)}$ is also closed under multiplication. Can you find a field K for which $D_{K(3)}$ is not closed?

Theorem. *$D_{K(2^m)}$ is closed under multiplication for every m.*

We will outline the proof using an inductive argument and some elementary matrix theory. There are many ways to generalize those ideas, and we will mention some of them.

$$a^2b^2 = (ab)^2$$

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Wednesday, October 22, 5 PM

Undergraduate Math Study Space (MA 052)

Free pizza!