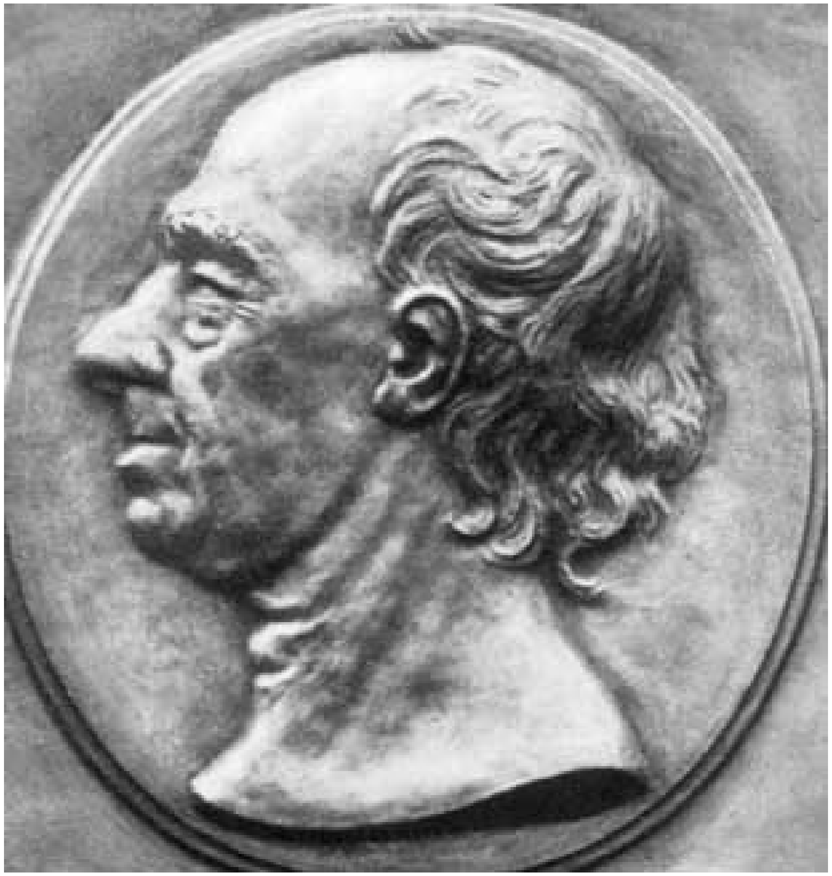


Euler’s Three Body Problem

In physics and astronomy, Euler’s three-body problem, named after Leonhard Euler, is to solve for the motion of a test mass that is free to move in the presence of the gravitational field of a primary and secondary mass which are fixed in space. This problem is the simplest three-body problem that retains physical significance. Euler discussed it in memoirs published in 1760.

The problem is analytically solvable but requires the evaluation of elliptic integrals. Numerical methods may be used, such as Runge-Kutta, to solve the resulting ordinary differential equations approximately and to gain some feel for the physics.

Euler Pseudoprime

An odd composite integer n is called an Euler pseudoprime to base a, if a and n are coprime, and

$$a^{(n-1)/2} \equiv \pm 1 \pmod{n}$$

The motivation for this definition is the fact that all prime numbers p satisfy the above equation which can be deduced from Fermat's little theorem. Fermat's theorem asserts that if p is prime, and coprime to a, then ap-1 = 1 (mod p). Suppose that p>2 is prime, then p can be expressed as 2q+1 where q is an integer. Thus; a(2q+1)-1 = 1 (mod p) which means that a2q - 1 = 0 (mod p). This can be factored as (aq - 1)(aq + 1) = 0 (mod p) which is equivalent to a(p-1)/2 = ±1 (mod p).

The equation can be tested rather quickly, which can be used for probabilistic primality testing. These tests are twice as strong as tests based on Fermat's little theorem.

Every Euler pseudoprime is also a Fermat pseudoprime. It is not possible to produce a definite test of primality based on whether a number is an Euler pseudoprime because there exist absolute Euler pseudoprimes, numbers which are Euler pseudoprimes to every base relatively prime to themselves. The absolute Euler pseudoprimes are a subset of the absolute Fermat pseudoprimes, or Carmichael numbers, and the smallest absolute Euler pseudoprime is 1729 = 7·13·19.

It should be noted that the stronger condition that a(n-1)/2 = (a/n) (mod n), where (a,n)=1 and (a/n) is the Jacobi symbol, is sometimes used for a definition of an Euler pseudoprime. A discussion of numbers of this form can be found at Euler-Jacobi pseudoprime.

Euler's phi-function

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

In number theory, the totient $\phi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n that are coprime to n. For example, $\phi(9) = 6$ since the six numbers 1, 2, 4, 5, 7 and 8 are coprime to 9. The function ϕ so defined is the totient function. The totient is usually called the Euler totient or Euler's totient, after the Swiss mathematician Leonhard Euler, who studied it. The totient function is also called Euler's phi function or simply the phi function, since it is commonly denoted by the Greek letter Phi (ϕ). The cototient of n is defined as $n - \phi(n)$, ie. the number of positive integers less than or equal to n that are not coprime to n.

The totient function is important mainly because it gives the size of the multiplicative group of integers modulo n. More precisely, $\phi(n)$ is the order of the group of units of the ring$\mathbb{Z}/n\mathbb{Z}$. This fact, together with Lagrange's theorem, provides a proof for Euler's theorem.

Euler's Formula in Graph Theory

Euler's formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely-large region), then

$$v - e + f = 2$$

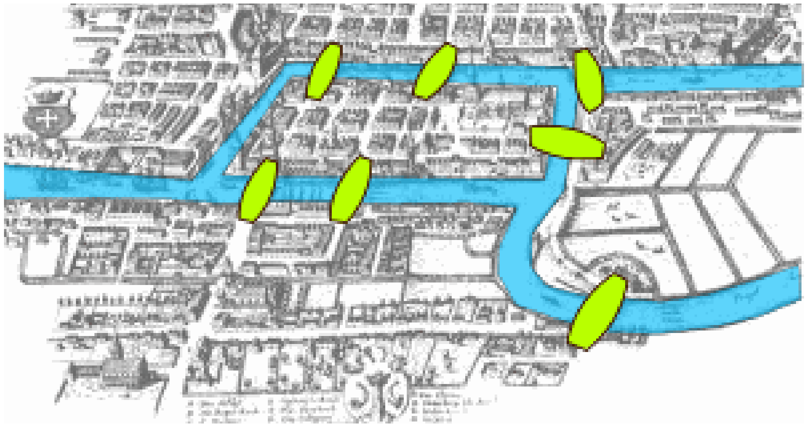
i.e. the Euler characteristic is 2. As an illustration, in the first planar graph given above, we have $v=6$, $e=7$ and $f=3$. If the second graph is redrawn without edge intersections, we get $v=4$, $e=6$ and $f=4$. Euler's formula can be proven as follows: if the graph isn't a tree, then remove an edge which completes a cycle. This lowers both e and f by one, leaving $v - e + f$ constant. Repeat until you arrive at a tree; trees have $v = e + 1$ and $f = 1$, yielding $v - e + f = 2$.

In a finite, connected, simple, planar graph, any face (except possibly the outer one) is bounded by at least three edges and every edge touches at most two faces; using Euler's formula, one can then show that these graphs are sparse in the sense that $e \leq 3v - 6$ if $v \geq 3$.

Note that Euler's formula is also valid for simple polyhedra. This is no coincidence: every simple polyhedron can be turned into a connected, simple, planar graph by using the polyhedron's vertices as vertices of the graph and the polyhedron's edges as edges of the graph. The faces of the resulting planar graph then correspond to the faces of the polyhedron. For example, the second planar graph shown above corresponds to a tetrahedron. Not every connected, simple, planar graph belongs to a simple polyhedron in this fashion: the trees do not, for example. A theorem of Ernst Steinitz says that the planar graphs formed from convex polyhedra (equivalently: those formed from simple polyhedra) are precisely the finite 3-connected simple planar graphs.

Eulerian Paths

In the mathematical field of graph theory, an Eulerian path is a path in a graph which visits each edge exactly once. Similarly, an Eulerian circuit is an Eulerian path which starts and ends on the same vertex. They were first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736. Graphs which allow the construction of so called Eulerian cycles are called Eulerian graphs. Euler observed that a necessary condition for the existence of Eulerian cycles is that all vertices in the graph have an even degree, and that for an Eulerian path either all, or all but two, vertices have an even degree; this means the Königsberg graph is not Eulerian.



Carl Hierholzer published the first complete characterization of Eulerian graphs in 1873, by proving that in fact the Eulerian graphs are exactly the graphs which are connected and where every vertex has an