Final exam **Math 5590H**

Due by Tuesday, December 12

50% **1.** Let $D \neq 0, 1$ be a square-free integer, and let $R = \mathbb{Z}[\omega]$ where $\omega = \sqrt{D}$ or $\omega = \frac{1+\sqrt{D}}{2}$ if $D = 1 \bmod 4$. Let $p \in \mathbb{N}$ be a prime.

(**a**) Prove that $R \cong \mathbb{Z}[x]/(f)$, where $f = x^2 - D$ if $\omega = \sqrt{D}$ and $f = x^2 - x + \frac{1-D}{4}$ if $\omega = \frac{1+\sqrt{D}}{2}$.

(**b**) Prove that $R/(p) \cong \mathbb{F}_p[x]/(f)$ (where $\mathbb{F}_p = \mathbb{Z}_p$).

(**c**) Prove that, under the isomorphism in (b), the conjugation automorphism of R acts on $\mathbb{F}_p[x]/(f)$ by $x \mapsto -x$ if $\omega = \sqrt{D}$ and $x \mapsto 1 - x$ if $\omega = \frac{1+\sqrt{D}}{2}$.

(**d**) Prove that

(i) if f has no roots in $\mathbb{F}_p$, p is inert (is prime) in R;

(ii) if f has two distinct roots in $\mathbb{F}_p$, then p splits in R into a product of two distinct maximal ideals: $(p) = P_1P_2$;

(iii) and if f has a double root (a root of multiplicity two) in $\mathbb{F}_p$, then p ramifies in R: $(p) = P^2$, where P is a maximal ideal in R.

(**e**) If p splits in R, $(p) = P_1P_2$, prove that P_1 and P_2 are conjugate, $P_2 = \overline{P}_1$; if p ramifies, $(p) = P^2$, then P is self-conjugate, $\overline{P} = P$.

(**f**) In the case $\omega = \sqrt{D}$, prove that p ramifies if and only if $p = 2$ or $p \mid D$.

(**g**) Prove that 7 is irreducible but not prime in $\mathbb{Z}[\sqrt{-5}]$, and that 11 is prime in $\mathbb{Z}[\sqrt{-5}]$.

10% **2.** Prove that the quotient ring $\mathbb{Z}[x]/(2x - 1)$ is isomorphic to the ring of fractions $D^{-1}\mathbb{Z}$ where $D = \{2^n,\ n \in \mathbb{N}\}$.

3. Let R be a PID.

15% (**a**) Prove that every prime ideal in $R[x]$ can be generated by at most two generators. (*Hint:* If P is a prime ideal in $R[x]$, in the case $P \cap R \neq 0$ consider the ring $R[x]/(P \cap R)$, and in the case $P \cap R = 0$, pass to $F[x]$ where F is the field of fractions of R. (You don't have to follow this hint though...))

5% (**b**) If S is a ring containing R and generated by a single element over R, $S = R[\alpha]$, prove that every prime ideal in S can be generated by at most two elements.

10% **4.** Prove that the polynmial $6x^5 - 55x^3 + 50x^2 + 15$ is irreducible over the field $\mathbb{Q}[i]$ (that is, in the ring $\mathbb{Q}[i][x]$).

10% **5.** A group N is said to be *complete* if the center of N is trivial and every automorphism of N is inner. Show that if G is a group, $N \trianglelefteq G$, and N is complete, then $G = N \times C_G(N)$.

15% **6.** Prove that no group of order $2004 = 2^2 \cdot 3 \cdot 167$ is simple. Give an example of a group of order 2004 in which a Sylow 3-subgroup is not normal.