Homework 1 **Math 5590H**

Due by Tuesday, August 29

A1. Let G be a semigroup with a left-neutral element e, that is, $ea = a$ for all $a \in G$.

(a) Suppose that every element in G has a left inverse with respect to e: for every $a \in G$ there exists $b \in G$ such that $ba = e$. Prove that G is a group.

(b) Show by example that it may be the case that every element in G has a right inverse with respect to e, but G is not a group.

A semigroup G is said to be *cancellative* if it has the cancellation properties: for any $a, b, c \in G$, if $ac = bc$ or $ca = cb$, then $a = b$.

A2. Prove that every finite cancellative semigroup is a group. (*Hint:* The cancellation properties say that for every $c \in G$, the left multiplication by c, $a \mapsto ca$, and the right multiplication by c, $a \mapsto ac$, are injective mappings $G \longrightarrow G$.)

A3. Let G be a group. Introduce the binary operation $*$ on G by $a * b = ba$.

(a) Prove that $(G, *)$ is a group.

(b) Prove that the mapping $a \mapsto a^{-1}$ defines an isomorphism between G and $(G, *)$.

A4. The symmetric group S_3 (the group of permutations of the set $\{1, 2, 3\}$) has 6 elements: $1 = \mathrm{Id}$, $\sigma\colon \left|\begin{smallmatrix}1\mapsto 2\\ 2\mapsto 3\\ 3\mapsto 1\end{smallmatrix}\right.$, $\sigma^2\colon \left|\begin{smallmatrix}1\mapsto 3\\ 2\mapsto 1\\ 3\mapsto 2\end{smallmatrix}\right.$, $\tau_1\colon \left|\begin{smallmatrix}1\mapsto 2\\ 2\mapsto 1\\ 3\mapsto 3\end{smallmatrix}\right.$, $\tau_2\colon \left|\begin{smallmatrix}1\mapsto 1\\ 2\mapsto 3\\ 3\mapsto 2\end{smallmatrix}\right.$, $\tau_3\colon \left|\begin{smallmatrix}1\mapsto 3\\ 2\mapsto 2\\ 3\mapsto 1\end{smallmatrix}\right.$. Write out the multiplication table for S_3 and find the orders of its elements. (Notice that S_3 is not commutative.)

1.1.25. If G is a group such that $a^2 = 1$ for all $a \in G$, prove that G is abelian.

1.1.31. Prove that any finite group G of even order contains an element of order 2.