Homework 10 **Math 5590H**

Due by Tuesday, November 7

The center $Z(R)$ of a ring R is the set of elements of R that commute with all other elements of R.

7.1.7. Prove that the center $Z(R)$ of a ring R is a subring of R (that is, is closed under subtraction and multiplication).

A2. Find all zero divisors in the ring $C(\mathbb{R})$ of continuous funcitons $\mathbb{R} \longrightarrow \mathbb{R}$. (Some real analysis is needed here, sorry.)

A nonzero element e of a ring R is said to be *idempotent* if $e^2 = e$.

A3. Let R be a commutative unital ring and let $e \in R$ be idempotent.

(a) Prove that $1 - e$ is also idempotent.

(b) Prove that $Re = \{ae,\ a \in R\}$ and $R(1-e)$ are subrings of R, that e is the identity in Re, $(1-e)$ is an identity in $R(1-e)$, and that $bc = 0$ for any $b \in Re$ and $c \in R(1-e)$.

(c) Prove that $R = Re \times R(1-e)$ as groups under addition, and for $a_1 = b_1e + c_1(1-e)$ and $a_2 = b_2e + c_2(1-e)$ one has $a_1a_2 = b_1b_2e + c_1c_2(1-e)$.

(d) Conversely, given two unital rings R_1 and R_2, show that the ring $R = R_1 \times R_2$ has an idempotent element e such that $R_1 = Re$ and $R_2 = R(1-e)$.

An element a of a ring R is said to be *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.

7.3.29. If R is a commutative ring, prove that the set $\mathrm{Nil}(R)$ of nilpotent elements of R is a subring of R. (*Hint:* To prove that $\mathrm{Nil}(R)$ is closed under addition, use the binomial formula (which is true in any commutative ring).)

7.1.14. Let x be a nilpotent element of a commutative ring R.

(a) Prove that x is either zero or a zero divisor.

(b) Prove that rx is nilpotent for all $r \in R$.

(c) If R is unital, prove that $1 + x$ is a unit in R. (This says that unipotent elements are units.) (*Hint:* Use the finite geometric progression summation formula.)

(d) If R is unital and a is a unit, prove that $a + x$ is a unit.