Homework 11 **Math 5590H**

Due by Tuesday, November 14

For a subset C of a ring R, *the left annihilator* of C is the set $\mathrm{Ann}(C) = \{a \in R : aC = 0\}$.

Cf. 7.3.22. Let R be a ring, C be a subset of R, and $A = \mathrm{Ann}(C)$.

(a) Prove that A is a left ideal in R.

(b) If C is a left ideal of R, prove that A is a two-sided ideal.

7.3.20. (a) If I is a left ideal in a ring R and S is a subring of R, prove that $I \cap S$ is a left ideal in S.

(b) Show by example that not every left ideal of a subring S of a ring R needs to be of the form $I \cap S$ for some left ideal I of R.

7.4.6. Prove that a unital ring R is a division ring iff it has no nontrivial ($\neq 0, R$) left ideals.

7.4.15. Let $x^2 + x + 1$ be an element of the polynomial ring $E = \mathbb{F}_2[x]$ and let $\overline{E} = E/(g)$ where $g = x^2 + x + 1$. For $f \in E$, let $\overline{f}$ be the image of f in $\overline{E}$.

(a) Prove that $\overline{E} = \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\}$.

(b) Write the 4×4 addition table for $\overline{E}$ and deduce that $(\overline{E}, +) \cong V_4$.

(c) Write the 4×4 multiplication table for $\overline{E}$ and deduce that $(\overline{E}^*, \cdot) \cong \mathbb{Z}_3$. Deduce that $\overline{E}$ is a field.

Let R be a commutative unital ring. For ideals I, J of R we say that I divides J, $I \mid J$, if $J \subseteq I$. We define $gcd(I, J) = I + J$ and $\mathrm{lcm}(I, J) = I \cap J$.

A1. Let I, J, L be ideals in R. Prove that

(a) if $I \,\big|\, J \,\big|\, L$ then $I \,\big|\, L$;

(b) if $I \,\big|\, J$ and $I \,\big|\, L$ then $I \,\big|\, \gcd(J, L)$;

(c) if $I \,\big|\, L$ and $J \,\big|\, L$, then $\mathrm{lcm}(I, J) \,\big|\, L$;

(d) $IJ \,\big|\, \gcd(I, J)\,\mathrm{lcm}(I, J)$.

(e) In the ring $\mathbb{Z}[x]$ (of polynomials with integer coefficients) let $I = (4)$ and $J = (2x)$. Prove that $IJ \neq \gcd(I, J)\,\mathrm{lcm}(I, J)$.

Let R be a commutative unital ring. A proper ideal M of R is said to be maximal if for any ideal L with $L \mid M$ we have $L = M$ or $L = R$. A proper ideal P of R is said to be prime if for any ideals I, J such that $P \mid IJ$ we have $P \mid I$ or $P \mid J$, which is equivalent to: for any $a, b \in R$ with $ab \in P$ we have $a \in P$ or $b \in P$.

Cf. 7.4.13. Let R be a commutative unital rings and S be a subring of R.

(a) If P is a prime ideal in R, prove that $P \cap S$ is either S or a prime ideal in S.

(b) Give an example of a ring R with a subring S and a maximal ideal M such that $M \cap S$ is neither S nor a maximal ideal of S.

7.4.33. (Some real analysis, sorry.) Let R be the ring $C([0,1])$ of continuous functions $f\colon [0,1] \longrightarrow \mathbb{R}$, and for each $c \in [0,1]$ let $M_c = \{f \in R \mid f(c) = 0\}$.

(a) Prove that if M is a maximal ideal in R then $M = M_c$ for some $c \in [0,1]$. (*Hint:* Assume that for every $c \in [0,1]$ there is $f_c \in M$ such that $f_c(c) \neq 0$. Using compactness of $[0,1]$, prove that M contains a unit.)

(d) Prove that, for $c \in [0,1]$, M_c is not finitely generated. (*Hint:* Given $f_1, \ldots, f_k \in M_c$, construct a function $f \in M_c$ which "tends to 0 at c slower than each of f_i", so that $f \neq g_1 f_1 + \cdots + g_k f_k$.)