Homework 12 **Math 5590H**

Due by Tuesday, November 21

In all problems, R is a commutative unital ring.

The intersection of all maximal ideals of R is called *the Jacobson ideal* of R, $\mathrm{Jac}(R)$.

A1. Prove that $a \in \mathrm{Jac}(R)$ iff $1 + ab$ is a unit for all $b \in R$. (*Hint:* $c \in R$ is a unit iff c is contained in no maximal ideal of R.)

A commutative unital ring is said to be *local* if it has single maximal ideal.

7.4.37. Let R be a commutative unital ring.

(a) Prove that if R is local with maximal ideal M, then $M = \{a \in R : a \text{ is not a unit}\}$. Conversely, prove that if the nonunit elements of R form an ideal M, then R is local with maximal ideal M.

(b) Let P be a prime ideal in R, let $D = R \setminus P$ (the difference, not the factor); then D is a multiplicative set. If D contains no zero divisors, prove that $D^{-1}R$ is a local ring.

$\mathrm{Spec}(R)$ is the set of all prime ideals of R. Closed sets in $\mathrm{Spec}(R)$ are subsets of the form $V_I = \{P : I \subseteq P\}$, where I are ideals in R.

A2. (a) Prove that the union of two closed subsets of $\mathrm{Spec}(R)$ is closed.

(b) Prove that the intersection of any collection of closed subsets of $\mathrm{Spec}(R)$ is closed.

A3. Let $\varphi: R \longrightarrow S$ be a homomorphism of commutative unital rings. Prove that the induced mapping $\eta: \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ (defined by $\eta(P) = \varphi^{-1}(P)$) is continuous, that is, for any closed subset V of $\mathrm{Spec}(R)$, $\eta^{-1}(V)$ is a closed subset of $\mathrm{Spec}(S)$. (*Hint:* Show that $\eta^{-1}(V) = V_J$ where J is the ideal of S generated by $\varphi(I)$.)

For an ideal I of R, its *radical* $\mathrm{rad}(I) = \{a \in R : a^n \in I \text{ for some } n\}$. An ideal I of R is said to be radical if $\mathrm{rad}(I) = I$, that is, $a^n \in I$ implies that $a \in I$. The radical of a primary ideal is prime, the converse is not true.

A4. If Q is an ideal of R such that the ideal $\mathrm{rad}(Q)$ is maximal, prove that Q is primary. (*Hint:* Show that R/Q is local, with $\mathrm{Nil}(R/Q)$ being the only maximal ideal.)

A5. If R is Noetherian, prove that any proper radical ideal in R is an intersection of finitely many prime ideals. (*Hint:* If the statement is false, let I be a maximal radical ideal not representable as a finite intersection of prime ideals. Let $a, b \in R \setminus I$ be such that $ab \in I$; show that $\mathrm{rad}(I + (a)) \cap \mathrm{rad}(I + (b)) = I$.)