Homework 13 **Math 5590H**

Due by Tuesday, December 5

8.2.4. Let R be an integral domain. Prove that the following two conditions (together) imply that R is a PID:

(i) Any two nonzero elements $a, b \in R$ have a greatest common divisor of the form $ra + sb$ for some $r, s \in R$.

(ii) R satisfies the ascending chain condition for principal ideals: if $a_1, a_2, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is n such that the elements $a_n, a_{n+1}, \ldots$ are all associate.

8.1.7(a). Find the generator for the ideal $(85, 1 + 13i)$ in $\mathbb{Z}[i]$.

8.1.9. Prove that the ring $\mathbb{Z}[\sqrt{2}]$ is a ED with respect to the norm $N(a+b\sqrt{2}) = |a^2 - 2b^2|$.

8.3.5. Let $R = \mathbb{Z}[\omega]$ where $\omega = \sqrt{-n}$ and n is a squarefree integer ≥ 5.

(a) Prove that 2 is irreducible in R.

(b) Prove that 2 is not prime in R and deduce that R is not a UFD. (*Hint:* Consider the cases of even and of odd n separately.)

(Note that in the case $n = 1 \, mod \, 4$ or $2 \, mod \, 4$, $\mathbb{Z}[\omega]$ is the ring of quadratic integers associated with $D = -n$, so all these rings are not UFDs.)

8.3.8. Let $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$, the ring of quadratic integers associated with $D = -5$. Let $\alpha = 1 + \sqrt{-5}$, then $\overline{\alpha} = 1 - \sqrt{-5}$.

(b) Let $I_2 = (2, \alpha)$ and $I_3 = (3, \alpha)$, then $\overline{I}_3 = (3, \overline{\alpha})$. Prove that $\overline{I}_2 = I_2$, and that I_2, I_3, and $\overline{I}_3$ are maximal ideals in $\mathcal{O}$.

(c) Prove that $(2) = I_2^2$, $(3) = I_3 \overline{I}_3$, $(\alpha) = I_2 I_3$, and $(\overline{\alpha}) = I_2 \overline{I}_3$. (This shows that $(6) = (2)(3) = (\alpha)(\overline{\alpha}) = I_2^2 I_3 \overline{I}_3$.)

8.3.9. If a quadratic integer ring $\mathcal{O}$ is a PID, prove that the absolute value $|N|$ of the field norm N on $\mathcal{O}$ is a Dedekind-Hasse norm.