Homework 2 **Math 5590H**

Due by Wednesday, September 6

Given a set X, S_X is the group of self-bijections $A \longrightarrow A$, with the operation of composition.

1.6.8,10. Let A and B be two sets of the same cardinality, that is, such that there exists a bijection $\varphi\colon A \longrightarrow B$. Define a mapping $\Phi\colon S_A \longrightarrow S_B$ by $\Phi(\sigma) = \varphi\circ\sigma\circ\varphi^{-1}$, $\sigma \in S_A$. Prove that Φ is an isomorphism between S_A and S_B.

2.1.4. (a) If G is a finite group and H is a nonempty subset of G such that $HH \subseteq H$, prove that H is a subgroup of G.

(b) Show by example that it may not be so if G is infnite.

2.1.6. (a) Let G be an abelian group. Prove that $H = \{g \in G : |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup* of G).

(b) Give an example where G is nonabelian and H is not a subgroup.

2.3.16. If x and y commute, prove that $|xy|$ divides $\mathrm{lcm}(|x|, |y|)$. Show that this may not be so if x and y do not commute. When x and y commute, show that $|xy|$ may not be equal to $\mathrm{lcm}(|x|, |y|)$.

1.2.3. Using the standard presentation $D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1}\rangle$ of the dihedral group, prove that all elements of D_{2n} of the form sr^k have order 2. Deduce that D_{2n} is generated by two elements, s and sr, of order 2.

1.6.24. If G is a finite group generated by two distinct elements a and b of order 2, prove that $G \cong D_{2n}$, where $n = |ab|$.

1.5.3. Using the generators i, j of Q_8, find a set R of relations to get a presentation of Q_8 in the form $Q_8 = \langle i, j \mid R\rangle$.

2.3.12. Prove that the following groups are not cyclic (cannot be generated by a single element):

(a) $\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) $\mathbb{Z}_2 \times \mathbb{Z}$.

(c) $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.