Homework 4 **Math 5590H**

Due by Tuesday, September 19

3.1.25,27. (a) Prove that a subgroup N of G is normal iff $aNa^{-1} \subseteq N$ for all $a \in G$. (*Hint:* To prove that $aNa^{-1} \supseteq N$, use the conjugation by a^{-1}.)

(b) Let $G = \mathrm{GL}_2(\mathbb{Q})$, let N be the subgroup of upper triangular matrices with integer entries and 1-s on the main diagonal: $N = \left\{\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right),\ k \in \mathbb{Z}\right\}$, and let $a = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. Show that $aNa^{-1} \subseteq N$ but $aNa^{-1} \neq N$.

(c) Let N be a finite subgroup of a group G. Prove that if $aNa^{-1} \subseteq N$ for some $a \in G$, then $aNa^{-1} = N$.

3.1.36. Prove that if $G/Z(G)$ is cyclic, then G is abelian. (*Hint:* If $\overline{x}$ is a generator of $G/Z(G)$, prove that every element of G can be written in the form $x^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.)

Cf. 3.1.39. If G is a non-abelian group, prove that the diagonal subgroup $D = \{(a, a) \mid a \in G\}$ of $G^2 = G \times G$ is not normal in G^2.

3.1.14(b). Show that each element of $\mathbb{Q}/\mathbb{Z}$ has a finite order but there are elements of arbitrarily large order.

3.1.34. Let $D_{2n} = \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle$ be the standard presentation of the dihedral group D_{2n} and let $k \mid n$.

(a) Prove that $H = \langle r^k \rangle$ is a normal subgroup of D_{2n}.

(b) Prove that $D_{2n}/H \cong D_{2k}$.

3.1.42. Assume that H and K are normal subgroups of G with $H \cap K = 1$. Prove that H and K commute: $xy = yx$ for all $x \in H$ and $y \in K$. (*Hint:* Show that $[x, y] = xyx^{-1}y^{-1} \in H \cap K$.)

3.2.5. Prove that if H is the only subgroup of order n in G, then $H \trianglelefteq G$. (*Hint:* For $a \in G$, notice that aHa^{-1} is a subgroup of G of the same order as H.)