Due by Tuesday, September 26

3.3.3. If H is a normal subgroup of a group G of prime index p, prove that for any $K \leq G$ either $K \leq H$, or $HK = G$ and $|K : (K \cap H)| = p$.

3.2.19. Prove that if N is a normal subgroup of a finite group G and $\gcd(|N|, |G : N|) = 1$ then N is the unique subgroup of G of order $|N|$. (*Hint:* If $K \subseteq G$ with $|K| = |N|$, consider the image of K under the projection $G \longrightarrow G/N$.)

3.3.7. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. (*Hint:* Consider the homomorphism $G \longrightarrow (G/M) \times (G/N)$, $a \mapsto (a \bmod M, a \bmod N)$.)

A1. Let F be a finite field of order q (that is, $F = \mathbb{F}_q$), let $n \in \mathbb{N}$, let N be the group $\{c \in F : c^n = 1\}$ of n-th roots of unity in F, let $|N| = d$. The special linear group $\mathrm{SL}_n(F)$ is the group of $n \times n$ matrices with determinant 1, the group $\mathrm{PSL}_n(F)$ is defined as $\mathrm{SL}_n(F)/Z\big(\mathrm{SL}_n(F)\big)$. Find the order of $\mathrm{SL}_n(F)$ and of $\mathrm{PSL}_n(F)$.

A2. Find a composition series for the groups
(a) Q_8. (b) D_8. (c) D_{12}.

A group G is said to be *solvable* if it is “made of abelian groups”: has a finite subnormal series with abelian factors, $1 = H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$ so that H_{i+1}/H_i is abelian for all i. A group G is said to be *polycyclic* if it is “made of cyclic groups”: has a finite subnormal series with cyclic factors, $1 = K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_n = G$ so that K_{j+1}/K_j is cyclic for all j. (Clearly, every polycyclic group is solvable.)

Cf. 3.4.8. Prove that a finite group is solvable iff it is polycyclic. Find (an infinite) solvable non-polycyclic group.

4.1.9. Assume that a group G acts transitively on the finite set X and let H be a normal subgroup of G. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the distinct orbits of H in X.

(a) Prove that G permutes the orbits $\mathcal{O}_i$: for every $a \in G$ and each i, $a\mathcal{O}_i = \mathcal{O}_j$ for some j. Prove that G acts transitively on the set $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$. Deduce that all the orbits $\mathcal{O}_i$ have the same cardinality.

(b) Let $x \in X$ and $\mathcal{O} = Hx$. Prove that $|\mathcal{O}| = \big|H : (H \cap G_x)\big|$ (where G_x is the stabilizer of x in G) and that $r = |G : HG_x|$.