Homework 6 **Math 5590H**

Due by Tuesday, October 3

A1. Let H be a subgroup of a group G. Prove that $N = \bigcap_{a\in G} aHa^{-1}$ is normal in G and is the maximal subgroup of H that is normal in G. (That is, if $K \leq H$ and $K \trianglelefteq G$, then $K \leq N$.)

A2. (a) If a group G act on a set X and N is the kernel of this action, show that the quotient group G/N also acts on X by $\overline{a}x = ax$, $x \in X$, $a \in G$.

(b) Let an action of a group G on a set X be transitive and such that for some $x \in X$ the stabilizer $N = G_x$ is a normal subgroup of G. Prove that N is the kernel of the action and that the induced action of G/N on X is regular. (An action is said to be regular if for any $x, y \in X$ there is a unique $a \in G$ such that $ax = y$.)

4.2.4. Use the left regular representation of the group Q_8 to find two elements of S_8 that generate a group isomorphic to Q_8.

4.2.5a. In the standard presentation for D_8, let $H = \langle s\rangle$. Enumerate the left cosets of H in G. Find the homomorphism $D_8 \longrightarrow S_4$ induced by the action of D_8 by left multiplications on the set D_8/H (of left cosets of H in D_8).

A3. How many elements of S_6 commute with $\sigma = (1, 2)(3, 4, 5)$?

A4. (a) If finite groups A and B have coprime orders, prove that any subgroup of $A \times B$ has form $H \times K$ where $H \leq A$ and $K \leq B$. (*Hint:* Consider the groups $H = \pi_1(G) \leq A$ and $K = \pi_2(G) \leq B$ where π_1 and π_2 are the projections from $A \times B$ onto A and B respectively. Prove that $|G| = |H| \cdot |K|$.)

(b) Give an example of a subgroup of $A \times B$, with $|A|$ and $|B|$ not coprime, NOT of the form $H \times K$ with $H \leq A$ and $K \leq B$.

A5. Prove that $\mathrm{GL}_2(\mathbb{R}) \neq \mathrm{SL}_2(\mathbb{R}) \times C$, whereas $\mathrm{GL}_3(\mathbb{R}) = \mathrm{SL}_3(\mathbb{R}) \times C$, where C is the subgroup of scalar matrices cI, $c \in \mathbb{R}^*$.