Homework 7 **Math 5590H**

Due by Tuesday, October 10

5.4.13. Prove that for any $n \in \mathbb{N}$, D_{8n} is not isomorphic to $D_{4n} \times \mathbb{Z}_2$.

For a sequence $H_1, H_2, \ldots$ of groups the direct product $\prod_{n=1}^{\infty} H_i$ is the group of sequences $(a_1, a_2, \ldots)$ with $a_n \in H_n$ for all n, and the direct sum $\bigoplus_{n=1}^{\infty} H_i$ is the subgroup of $\prod_{n=1}^{\infty} H_i$ of sequences of the form $(a_1, a_2, \ldots, a_k, 1, 1, 1, \ldots)$.

A1. Prove that every element of the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}_n$ has finite order, and find an element of $\prod_{n=1}^{\infty} \mathbb{Z}_n$ of an infinite order. (Notice that since the groups are additive, the elements of $\bigoplus_{n=1}^{\infty} \mathbb{Z}_n$ have form $(a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots)$.)

Given groups H, K, and N with injective homomorphisms $\varphi\colon N \longrightarrow Z(H)$ and $\psi\colon N \longrightarrow Z(K)$, the (external) central product $H *_N K$ is the quotient group $(H \times K)/D$ where $D = \left\{(\varphi(a), \psi(a)^{-1}),\ a \in N\right\}$.

A2. Let subgroups $H, K \leq G$ satisfy $HK = G$ and $hk = kh$ for all $h \in H$ and $k \in K$, and let $N = H \cap K$. Prove that $G \cong H *_N K$ under an isomorphism that "respects" H and K: $h \leftrightarrow (h, 1)$ and $k \leftrightarrow (1, k)$.

A3. Let $n, m \in \mathbb{N}$, and let $d = \gcd(n, m)$ and $l = \mathrm{lcm}(n, m)$. Prove that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_l \times \mathbb{Z}_d$. (*Hint:* Don't do anything complicated, just consider the prime factorizations of n and m, and then use the Chinese remainder theorem to disassamble and then reassemble this group.)

Given groups H, K, and N with surjective homomorphisms $\varphi\colon H \longrightarrow N$ and $\psi\colon K \longrightarrow N$, the relative direct product $H \times_N K$ is the subgroup $\left\{(h, k) \in H \times K : \varphi(h) = \psi(k)\right\}$ of $H \times K$.

A4. Let $n, m \in \mathbb{N}$, $d = \gcd(n, m)$, $l = \mathrm{lcm}(n, m)$. Then $\mathbb{Z}_d$ is a common factor of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Prove that $\mathbb{Z}_n \times_{\mathbb{Z}_d} \mathbb{Z}_m \cong \mathbb{Z}_l$.

5.2.2,3(a,b,c). Give the list of elementary divisors and the invariant factors of all abelian groups of the order:

(a) $270 = 2 \cdot 3^3 \cdot 5$ (b) $9801 = 3^4 \cdot 11^2$ (c) $320 = 2^6 \cdot 5$

5.2.4(b). Determine which pairs of abelian groups listed are isomorphic (where the expression $[n_1, \ldots, n_k]$ denotes the group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$):

$[2^2, 2 \cdot 3^2]$, $[2^2 \cdot 3, 2 \cdot 3]$, $[2^3 \cdot 3^2]$, $[2^2 \cdot 3^2, 2]$.