Homework 8 **Math 5590H**

Due by Wednesday, October 18

A1. We have $\mathrm{Aut}(D_8) = \{\varphi_{k,l},\ k \in \mathbb{Z}_4^*,\ l \in \mathbb{Z}_4\}$, where $\varphi_{k,l} : \left| \begin{smallmatrix} r \mapsto r^k \\ s \mapsto sr^l \end{smallmatrix} \right.$. Find the group $\mathrm{Inn}(D_8)$ of inner autormorphisms of D_8 and find an outer automorphism of D_8.

Any automorphism $\varphi \in \mathrm{Aut}(Q_8)$ is uniquely defined by the elements $\varphi(i)$ and $\varphi(j)$. $\varphi(i)$ and $\varphi(j)$ can be any of the elements $\pm i, \pm j, \pm k$ with the only condition that $\varphi(i) \neq \pm\varphi(j)$. (Under this condition $\varphi(i)$ and $\varphi(j)$ generate Q_8 and satisfy the same relations as i and j.) Hence, $|\mathrm{Aut}(Q_8)| = 24$.

A2. Prove that $\mathrm{Aut}(Q_8) \cong S_4$. (*Hint:* Label the faces of a cube with the symbols $i, j, k, -i, -j, -k$ so that for any $x \in \{i, j, k\}$, x and $-x$ are located on the opposite faces. Prove that $\mathrm{Aut}(Q_8)$ is isomorphic to the group of rotations (rigid motions) of the cube. (Which, as we've already known, is $\cong S_4$.))

4.4.8. Let $H \leq K \leq G$.

(a) Prove that if $H \operatorname{char} K$ and $K \operatorname{char} G$, then $H \operatorname{char} G$. Use this result to prove that the subgroup $V = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is characteristic in S_4.

(b) Prove that if $H \operatorname{char} K$ and $K \trianglelefteq G$ then $H \trianglelefteq G$.

(c) Give an example to show that if $H \trianglelefteq K$ and $K \operatorname{char} G$ then H need not be normal in G.

4.4.13. Let G be a group of order $203 = 7 \cdot 29$ and H be a normal subgroup of G of order 7. Prove that $H \leq Z(G)$. Deduce that G is abelian. (*Hint:* For the first part, consider the action of G on H by conjugations, and prove that this action must be trivial. For the second part, notice that G/H, if nontrivial, is cyclic.)

A3. Construct a nonabelian group of order 39; give a presentation of this group in terms of generators and relations. (*Hint:* Construct a non-direct semidirect product of $H = \langle a \rangle \cong \mathbb{Z}_{13}$ and $K = \langle b \rangle \cong \mathbb{Z}_3$.)

5.5.9. The matrix $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 4 \end{smallmatrix}\right)$ has order 5 in $\mathrm{GL}_2(\mathbb{Z}_{19})$. Use it to construct a nonabelian group of order $1805 = 19^2 \cdot 5$. (*Hint:* Construct a non-direct semidirect product of $H = \langle a, b \rangle \cong \mathbb{Z}_{19}^2$ and $K = \langle c \rangle \cong \mathbb{Z}_5$, with the action of c on H by conjugations defined by the matrix A.)