Homework 9 **Math 5590H**

Due by Wednesday, October 25

A1. Let $p \geq 3$ be a prime.

(a) Prove that a nonabelian semidirect product $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ exists and is unique up to isomorphism. (*Hint:* Notice that $\mathrm{Aut}(\mathbb{Z}_{p^2})$ has a unique subgroup of order p.)

(b) Prove that a nonabelian semidirect product $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$ exists and is unique up to isomorphism. (*Hint:* Notice that the subgroups of order p in $\mathrm{Aut}(\mathbb{Z}_p^2)$ are Sylow, and so, are all conjugate.)

(c) Prove that nonabelian $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ exist and are both isomorphic to D_8.

4.5.7. Exhibit all Sylow 2-subgroups of S_4 and determine their isomorphism type.

4.5.13. Prove that every group of order 56 has a normal Sylow p-subgroup for some p. (*Hint:* If $n_7 \neq 1$, how many elements of order 7 are there in G?)

4.5.17, 6.2.15. (a) Prove that if $|G| = 105$ then G has a normal Sylow 5 subgroup and a normal Sylow 7-subgroup. (*Hint:* Counting elements, show that at least one of n_5 and n_7 is equal to 1. If $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$, prove that $PQ = P \times Q$ and is normal in G.)

(b) Find all (up to isomorphism) groups of order 105. (*Hint:* $16^3 = 1 \, mod \, 35$.)

A2. Prove that all groups of order p^2q^2, where p and q are prime, are solvable. (*Hint:* Assuming $q > p$, in the case $n_q \neq 1$, show that $|G| = 36$, in which case G is known to be solvable.)