Midterm review problems **Math 5590H**

1. If in a group G, $(ab)^{-1} = a^{-1}b^{-1}$ for al $a, b \in G$, prove that G is abelian.

2. Draw the lattice of subgroups of the groups S_3, D_8, Q_8. Determine which of these subgroups are normal and which are conjugate.

3. Describe all subgroups of $\mathbb{Z}$. Which of these subgroups are $6\mathbb{Z} \cap 8\mathbb{Z}$ and $6\mathbb{Z} + 8\mathbb{Z}$?

4. List all subgroups of $\mathbb{Z}_{12}$ and determine their isomorphism type.

5. How many elements generate the cyclic group $\mathbb{Z}_{20}$?

6. Prove that for any $a, n \in \mathbb{N}$ with $\gcd(a, n) = 1$ one has $a^{\varphi(n)} = 1 \bmod n$ (where $\varphi(n)$ is the Euler totient function).

7. (a) Show that $\mathbb{Z}_9^* \cong \mathbb{Z}_6$. (b) Show that $\mathbb{Z}_{12}^* \cong V_4$.

8. Let $H, K \leq G$ with $|H| = |K| = 6$. How many elements may the set HK have?

9. If $H, K \leq G$, prove that $|G : (H \cap K)| \leq |G : H| \cdot |G : K|$, and if $\big|G : (H \cap K)\big| = |G : H| \cdot |G : K|$, prove that $HK = G$.

10. If G and H are two groups of co-prime orders and $\varphi\colon G \longrightarrow H$ is a homomorphism, prove that φ is trivial, $\varphi(G) = 1$.

11. If G is a finite group such that $p = |G|$ is a prime integer, prove that G is a simple group and $G \cong \mathbb{Z}_p$.

12. If G is a solvable group of order n and $n = p_1 \cdots p_k$ is a prime factorization of n, explain why the factors of any composition series of G are (isomorphic to) $\mathbb{Z}_{p_1}, \ldots, \mathbb{Z}_{p_k}$.

13. Let $\varphi\colon G \longrightarrow H$ be a group homomorphism.

(a) If $N \trianglelefteq G$, is $\varphi(N)$ normal in H? In $\varphi(G)$?

(b) If $K \trianglelefteq H$, is $\varphi^{-1}(K)$ normal in G?

14. (a) If $H \trianglelefteq K \trianglelefteq G$, is it true that $H \trianglelefteq G$?

(b) If $H \leq K \leq G$ and $H \trianglelefteq G$, is it true that $H \trianglelefteq K$?

15. (a) If $H \leq G$ and $K \trianglelefteq G$, prove that the subgroup $H \cap K$ is normal in H.

(b) If $H, K \trianglelefteq G$, prove that the subgroup HK is normal in G.

16. (a) Give an example of a group G and its quotient group K such that K is not isomorphic to any subgroup of G.

(b) Give an example of a group G and its subgroup H such that H is not isomorphic to any quotient group of G.

17. Suppose that $H, K \trianglelefteq G$ and $H \leq K$. How are the quotient groups G/H and G/K related – is G/H isomorphic to a subgroup of G/K, G/K to a subgroup of G/H, G/H to a quotient group of G/K, or G/K to a quotient group of G/H?

18. If $H \leq G$ and $|G : H| = 2$, prove that $H \trianglelefteq G$.

19. If every subgroup of a group G is normal in G, does this imply that G is abelian?

20. If $\varphi\colon G \longrightarrow H$ is a group homomorphism and $a \in G$ is an element of finite order, prove that $|\varphi(a)| \,\big|\, |a|$.

21. Let G be a group and $N \trianglelefteq G$.

(a) If G is generated by a set of n elements, prove that G/N is also generated by a set of n elements.

(b) If G is generated by a set of n elements, show by example that N may not have a generating set of n elements.

(c) If N is generated by a set of m elements and G/N is generated by a set of k elements, prove that G is generated by a set of $m+k$ elements.

22. If G is a finite group, p is the minimal prime dividing $|G|$, and $H \leq G$, $|G : H| = p$, prove that $H \lhd G$.

23. Let H be a normal subgroup of a group G and assume that $G/H \cong V_4$. Draw the sub-lattice of the lattice of subgroups of G consisting of the subgroups that contain H.

24. Prove that the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle $S = \{z \in \mathbb{C}^* : |z| = 1\}$.

25. Prove that the mapping $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longrightarrow \varphi(z) = \frac{az+b}{cz+d}$ is a homomorphism from the group $\mathrm{GL}_2(\mathbb{C})$ onto the group $\mathcal{M}$ of Möbius (linear-fractional) transformations of the Riemann sphere. Prove that $\mathcal{M} \cong \mathrm{GL}_2(\mathbb{C})/Z\big(\mathrm{GL}_2(\mathbb{C})\big)$.

26. Present the group Q_8 as a quotient group of the free group F_n for some n.

27. Represent the group Q_8 as a subgroup of S_8.

28. Assume that groups G and H are defined by generators and relations, $G = \langle S|R\rangle$ and $H = \langle S|R'\rangle$, with $R \subseteq R'$. (That is, H has the same generators as G and all relations of G.) How are G and H related – is H isomorphic to a subgroup of G, G to a subgroup of H, H to a quotient group of G, or G to a quotient group of H?

29. Suppose that a group G acts transitively on a set X, and assume that the stabilizer G_x of a point $x \in X$ contains a nontrivial subgroup H which is normal in G. Prove that the action of G is not faithful.

30. Let $H \leq G$, and let φ be the action of G on G/H (the set of left cosets of H in G) by left multiplications: $a \cdot (bH) = abH$.

(a) For $b \in G$, what is the stabilizer G_{bH} of the coset $bH \in G/H$ under φ?

(b) Let K be a subgroup of H which is normal in G. Prove that K is contained in the kernel of φ.

31. If $|G : Z(G)| = m$, prove that the cardinality of any conjugacy class in G is a divisor of m.

32. Find a representative for each conjugacy class in S_6. Which of them are even and which are odd permutations?

33. If φ is a homomorphism $S_n \longrightarrow S_m$, explain why it cannot be that

(a) $\varphi((a,b)) = (x,y,z)$; (b) $\varphi((a,b,c)) = (x,y,z)$ and $\varphi((d,e,f)) = (u,v,w)(p,r,s)$. (It is assumed that different letters stand for distinct elements of the corresponding sets.)

34. Let $\sigma_1 = (1,2,3)(4,5)$ and $\sigma_2 = (1,3,5)(2,4)$. Find a permutation ρ such that $\rho\sigma_1\rho^{-1} = \sigma_2$.

35. Determine how many elements of S_7 commute with the permutation $(1,2,3)(4,5)(6,7)$.

36. Prove that $\mathrm{Hol}(V_4) \cong S_4$.

37. If H and K are normal subgroups of a finite group G such that $\gcd(|H|, |K|) = 1$ and $|H| \cdot |K| = |G|$, prove that $G = H \times K$.

38. Find all, up to isomorphism, abelian groups of order 720.

39. Determine whether the groups $\mathbb{Z}_{36} \times \mathbb{Z}_4 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{20} \times \mathbb{Z}_{72}$ are isomorphic.

40. Find the invariant factors of the group $\mathbb{Z}_{315}^*$.

41. Prove that $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

42. Find $\mathrm{Aut}(D_{10})$, $\mathrm{Inn}(D_{10})$, $\mathrm{Out}(D_{10})$.

43. Let p be a prime; find $\mathrm{Aut}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$.

44. Give an example of a group G with normal subgroup H such that $G \not\cong H \rtimes (G/H)$.

45. The matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}\right)$ has order 30 in the group $\mathrm{GL}_2(\mathbb{F}_{11})$. Construct a group of order $11^2 \cdot 30$.

46. Explain why the group $\langle a, b \mid a^{10} = b^3 = 1,\ ba = a^3 b \rangle$ is a semidirect product $\langle a \rangle \rtimes \langle b \rangle$, but is not isomorphic to a semidirect product $\mathbb{Z}_{10} \rtimes \mathbb{Z}_3$.

47. If H is a subgroup of a group G such that H is characteristic in $N_G(H)$, prove that $N_G(N_G(H)) = N_G(H)$.

48. If all Sylow subgroups of a finite group G are normal, prove that G is a direct product of its Sylow subgroups.

49. If G is a finite group, p is a prime divisor of $|G|$, and P_1, P_2 are distinct Sylow p-subgroups of G, prove that $P_2 \not\leq N_G(P_1)$.

50. If a Sylow subgroup P of a finite group G is normal, explain why P is characteristic in G.

51. Let R be a normal p-subgroup of a finite group G.

(a) Prove that R is contained in every Sylow p-subgroup of G.

(b) If S is another normal p-subgroup of G, prove that RS is also a normal p-subgroup of G.

(c) The subgroup $O_p(G)$ is defined to be the group generated by all normal p-subgroups of G. Prove that $O_p(G)$ is the largest normal p-subgroup of G and is equal to the intersection of all Sylow p-subgroups of G.

(d) Let $\overline{G} = G/O_p(G)$. Prove that $O_p(\overline{G}) = 1$.

52*. (Sorry, too difficult, forget it.) If G is a simple group of order p^2qr, where $p < q < r$ are primes, prove that $|G| = 60$.

53. If $P \in \mathrm{Syl}_p(G)$, explain why $|N_G(P)| = |G|/n_p$.

54. If G is a simple finite group of order n and p is a prime divisor of n, explain why G is isomorphic to a subgroup of A_{n_p}.

55. (a) If H is a subgroup of a finite group G, prove that $n_p(H) \leq n_p(G)$ for all prime p.

(b) If K is a quotient group of a finite group G, prove that $n_p(K) \leq n_p(G)$ for all prime p.

56. Prove that every group of order 77 is cyclic.

57. Prove that every group of order 45 is abelian.

58. How many elements of order 7 are there in a simple group of order 168?

59. Find all, up to isomorphism, groups of order 55.

60. Find all, up to isomorphism, groups of order 75.

61. Prove that there are no simple groups of order 72.

62. Prove that there are no simple groups of order 112.

63. Prove that every group of order 231 contains an element of order 77.

64. Find the derived series for S_4 and for S_5.

65. Find the upper and the lower central series for A_4 and for S_4.

66. If H and K are solvable subgroups of a group G with $K \leq N_G(H)$, prove that their join HK is solvable.

67. Prove that any group of order $5^5 \cdot 7^3$ is nilpotent.