Solutions to Homework 1 **Math 5590H**

A1. *Let G be a semigroup with a left-neutral element e, that is, $ea = a$ for all $a \in G$.*

10pt (a) *Suppose that every element in G has a left inverse with respect to e: for every $a \in G$ there exists $b \in G$ such that $ba = e$. Prove that G is a group.*

Solution. Let $a \in G$ and let b be a left inverse of a, $ba = e$; I'll show that b is a right inverse of a as well. Indeed, we have

$$(ab)(ab) = a(ba)b = aeb = ab;$$

cancelling ab (multiplying both parts by the inverse of ab from the left), we get that $ab = e$.

Now, let's show that e is a right neutral element in S. Indeed, for any $a \in G$,

$$ae = aa^{-1}a = ea = a.$$

5pt (b) *Show by example that it may be the case that every element in G has a right inverse with respect to e, but G is not a group.*

Solution. Let G be a set with ≥ 2 elements with operation defined by $ab = b$ for all $a, b \in G$. Every element of G is left-neutral; left e be one of them. Then for every $a \in S$, $ae = e$, so e is the right inverse of a. Clearly, G is not a group (it has no right neutral elements).

10pt **A2.** *Prove that every finite cancellative semigroup is a group.*

Solution. Let G be a cancellative semigroup. This means that for any $a \in G$, the left multiplication by a, $L_a(b) = ab$, and the right multiplication by a, $R_a(b) = ba$, are injective mappings: $ab_1 = ab_2$ implies that $b_1 = b_2$, and $b_1 a = b_2 a$ implies that $b_1 = b_2$. Since G is finite, this implies that, for any $a \in G$, these mappings are also surjective: for any $c \in G$ there exists $b \in G$ such that $ab = c$ and $d \in G$ such that $da = c$.

Fix any $a \in G$ and find $e \in G$ such that $ae = a$. Then for any $b \in G$, $aeb = ab$, and since G is cancellative, $eb = b$. Hence, e is a left neutral element in G. Similarly, we obtain that G has a right neutral element, so has a unique neutral element e.

Now, for any $a \in G$ there exists $b \in G$ such that $ab = e$ and there is $d \in G$ such that $da = e$, so a has a left and a right inverses. So, a has an inverse, and G is a group.

A3. *Let G be a group. Introduce the binary operation $*$ on G by $a * b = ba$.*

10pt (a) *Prove that $(G, *)$ is a group.*

Solution. First of all, $*$ is associaitve: for any $a, b, c \in G$,

$$(a * b) * c = (ba) * c = c(ba) = (cb)a = a * (cb) = a * (b * c).$$

With respect to $*$, the identity 1 of G is still the identity: for any $a \in G$, $1 * a = a1 = a$ and $a * 1 = 1a = a$. And for any $a \in G$, $a * (a^{-1}) = a^{-1}a = 1$, $a^{-1} * a = aa^{-1} = 1$, so, with respect to $*$, a^{-1} is still an inverse of a.

5pt (b) *Prove that the mapping $a \mapsto a^{-1}$ defines an isomorphism between G and $(G, *)$.*

Solution. If G is nonabelian, the identity mapping $G \longrightarrow G$, $a \mapsto a$, is not an isomorphism of groups G and $(G, *)$! (For $a, b \in G$, $ab \mapsto ab \neq a*b = ba$, generally speaking.) Consider, however, the mapping $\varphi(a) = a^{-1}$. φ is bijective. (It is the inverse of itself, $\varphi^{-1} = \varphi$, since $(a^{-1})^{-1} = a$ for all $a \in G$.) And for any $a, b \in G$, $\varphi(ab) = (ab)^{-1} = b^{-1}a^{-1} = \varphi(b)\varphi(a) = \varphi(a) * \varphi(b)$.

10pt **A4.** *The symmetric group S_3 (the group of permutations of the set $\{1, 2, 3\}$) has 6 elements:* $1 = \mathrm{Id}$, $\sigma\colon\left|\begin{smallmatrix}1\mapsto 2\\ 2\mapsto 3\\ 3\mapsto 1\end{smallmatrix}\right.$, $\sigma^2\colon\left|\begin{smallmatrix}1\mapsto 3\\ 2\mapsto 1\\ 3\mapsto 2\end{smallmatrix}\right.$, $\tau_1\colon\left|\begin{smallmatrix}1\mapsto 2\\ 2\mapsto 1\\ 3\mapsto 3\end{smallmatrix}\right.$, $\tau_2\colon\left|\begin{smallmatrix}1\mapsto 1\\ 2\mapsto 3\\ 3\mapsto 2\end{smallmatrix}\right.$, $\tau_3\colon\left|\begin{smallmatrix}1\mapsto 3\\ 2\mapsto 2\\ 3\mapsto 1\end{smallmatrix}\right.$. *Write out the multiplication table for S_3 and find the orders of its elements. (Notice that S_3 is not commutative.)*

Solution. The multiplication table of S_3 is

	1	τ_1	τ_2	τ_3	σ	σ^2
1	1	τ_1	τ_2	τ_3	σ	σ^2
τ_1	τ_1	1	σ	σ^2	τ_2	τ_3
τ_2	τ_2	σ^2	1	σ	τ_3	τ_1
τ_3	τ_3	σ	σ^2	1	τ_1	τ_2
σ	σ	τ_3	τ_1	τ_2	σ^2	1
σ^2	σ^2	τ_2	τ_3	τ_1	1	σ

(Recall that for two permutations φ and ψ, $\varphi\psi = \varphi\circ\psi$, that is, "$\psi$ acts first", $(\varphi\psi)(x) = \varphi(\psi(x))$.) Clearly, $\tau_1^2 = \tau_2^2 = \tau_3^2 = 1$ and $\sigma\sigma^2 = \sigma^3 = 1$. To compute, say, $\tau_1\tau_2$, notice that $(\tau_1\tau_2)(1) = \tau_1(\tau_2(1)) = \tau_1(1) = 2$. Since $\tau_1\tau_2 \neq \tau_1$, the only option is $\tau_1\tau_2 = \sigma$. Similarly, $\tau_2\tau_1(1) = 3$, so $\tau_2\tau_1 = \sigma^2$. Next, $\tau_1\sigma = \tau_1\tau_1\tau_2 = \tau_2$; or: $\tau_1\sigma(1) = 1$ and $\sigma \neq \tau_1^{-1}$, so $\tau_1\sigma = \tau_2$. And so on.

The orders of 1 is 1, of σ and σ^2 is 3, and of τ_1, τ_2, τ_3 is 2.

5pt **1.1.25.** *If G is a group such that $a^2 = 1$ for all $a \in G$, prove that G is abelian.*

Solution. For any $a \in G$, since $aa = 1$, we have $a^{-1} = a$. In particular, for any $a, b \in G$ we have $(ab)^{-1} = ab$; but also $(ab)^{-1} = b^{-1}a^{-1} = ba$, so $ba = ab$.

5pt **1.1.31.** *Prove that any finite group G of even order contains an element of order* 2.

Solution. Let $A = \{b \in G : b^{-1} \neq b\}$. Then A is partitioned into pairs of inverses, $\{b, b^{-1}\}$, so $|A|$ is even. Hence, $G \setminus A$ also has an even number of elements. One of these elements is 1, so there must be at least one more element $a \neq 1$ in $G \setminus A$, for which $a^{-1} = a$. For such a we have $|a| = 2$.