5pt **7.1.7.** *Prove that the center $Z(R)$ of a ring R is a subring of R (that is, is closed under subtraction and multiplication).*

Solution. If $a, b \in Z(R)$, then for any $c \in R$ we have $c(a-b) = ca - cb = ac - bc = (a-b)c$ and $c(ab) = acb = abc$, so $a - b, ab \in Z(R)$.

10pt **A2.** *Find all zero divisors in the ring $C(\mathbb{R})$ of continuous funcitons $\mathbb{R} \longrightarrow \mathbb{R}$.*

Solution. I claim that a nonzero function $f \in C(\mathbb{R})$ is a zero divisor in $C(\mathbb{R})$ iff there is an open interval $(a,b) \subset \mathbb{R}$ such that $f|_{(a,b)} = 0$. Indeed, if f is such a fnction, put $g(x) = 0$ if $x \leq a$ or $x \geq b$, $g(x) = \left|x - \frac{a+b}{2}\right| - \frac{b-a}{2}$; then $g \in C(\mathbb{R})$, $g \neq 0$ and $fg = 0$, so f is a zero divisor.

Conversely, if there is a nonzero $g \in C(\mathbb{R})$ such that $fg = 0$. Let $x_0 \in \mathbb{R}$ be such that $g(x_0) \neq 0$. Find $\varepsilon > 0$ such that $g(x) \neq 0$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Then, since $fg = 0$, we must have $f(x) = 0$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

A3. *Let R be a commutative unital ring and let $e \in R$ be idempotent.*

5pt (a) *Prove that $1 - e$ is also idempotent.*

Solution. $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$.

10pt (b) *Prove that $Re = \{ae,\ a \in R\}$ and $R(1-e)$ are subrings of R, that e is the identity in Re, $(1-e)$ is an identity in $R(1-e)$, and that $bc = 0$ for any $b \in Re$ and $c \in R(1-e)$.*

Solution. For any $a, b \in Re$ we have $a = a'e$, $b = b'e$ for some $a', b' \in R$, then $a - b = (a' - b')e \in Re$ and $ab = a'eb'e = a'b'e^2 = a'b'e \in Re$, so Re is a subring. Since $1 - e$ is also idempotent, $R(1-e)$ is a subring too.

For any $a \in Re$, $a = a'e$, we have $ae = a'e^2 = a'e = a$, so e is the identity of Re. Since $1 - e$ is also an idempotent, $1 - e$ is an identity in $R(1-e)$.

$e(1-e) = e - e^2 = e - e = 0$. Thus for any $b \in Re$ and $c \in R(1-e)$, $b = b'e$ and $c = c'(1-e)$, we have $bc = b'c'e(1-e) = 0$.

5pt (c) *Prove that $R = Re \times R(1-e)$ as groups under addition, and for $a_1 = b_1e + c_1(1-e)$ and $a_2 = b_2e + c_2(1-e)$ one has $a_1a_2 = b_1b_2e + c_1c_2(1-e)$.*

Solution. If $a \in Re \cap R(1-e)$ then $ae = a$ and $a(1-e) = a$, so $a = a(1-e) = ae(1-e) = 0$. Hence, $Re \cap R(1-e) = 0$. Also, for every $a \in R$, $a = ae + a(1-e)$, so $Re + R(1-e) = R$. Hence, $R = Re \times R(1-e)$. And for $a_1 = b_1e + c_1(1-e)$ and $a_2 = b_2e + c_2(1-e)$,

$$a_1a_2 = b_1b_2e^2 + b_1c_2e(1-e) + c_1b_2(1-e)e + c_1c_2(1-e)^2 = b_1b_2e + c_1c_2(1-e).$$

5pt (d) *Conversely, given two unital rings R_1 and R_2, show that the ring $R = R_1 \times R_2$ has an idempotent element e such that $R_1 = Re$ and $R_2 = R(1-e)$.*

Solution. Let $1 = (1,1) \in R$, then 1 is the identity of R. Let $e = (1,0) \in R$, then $1 - e = (0,1)$. R_1 is identified with $R_1 \times 0$, $a \leftrightarrow (a,0)$, and R_2 is identified with $0 \times R_2$, $b \leftrightarrow (0,b)$. Then $R_1 = \{(a,0),\ a \in R_1\} = Re$ and $R_2 = \{(0,b),\ b \in R_2\} = R(1-e)$.

10pt **7.3.29.** *If R is a commutative ring, prove that the set $\mathrm{Nil}(R)$ of nilpotent elements of R is a subring of R.*

Solution. If $a, b \in \mathrm{Nil}(R)$, with $a^n = 0$, then $(ab)^n = a^nb^n = 0$, so $ab \in \mathrm{Nil}(R)$. If $a, b \in \mathrm{Nil}(R)$, with $a^n = b^m = 0$, then by the binomial formula

$$(a+b)^{n+m-1} = \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} a^k b^{n+m-1-k}.$$

The form of the coefficients doesn't actually matter; what matters is that in each summand either $k \geq n$ and then $a^k = 0$, or $n + m - 1 - k \geq m$ and then $b^{n+m-1-k} = 0$. Hence, $(a+b)^{n+m-1} = 0$, and $a + b \in \mathrm{Nil}(R)$.

7.1.14. *Let x be a nilpotent element of a commutative ring R.*

5pt (a) *Prove that x is either zero or a zero divisor.*

Solution. If $x \neq 0$, let n be the minimal positive integer for which $x^n = 0$, then $n \geq 2$. Then $xx^{n-1} = 0$, where both $x, x^{n-1} \neq 0$, so x is a zero divisor.

5pt (b) *Prove that rx is nilpotent for all $r \in R$.*

Solution. If $x^n = 0$, then $(rx)^n = r^n x^n = 0$ (since R is commutative).

5pt (c) *If R is unital, prove that $1 + x$ is a unit in R. (This says that unipotent elements are units.)*

Solution. If $x^n = 0$, then $(1 + x)(1 - x + x^2 - \cdots \pm x^{n-1}) = 1 \pm x^n = 1$.

5pt (d) *If R is unital and a is a unit, prove that $a + x$ is a unit.*

Solution. $a + x = a(1 + a^{-1}x)$; $a^{-1}x$ is nilpotent, so $1 + a^{-1}x$ is a unit, and the product of units is a unit.