In all problems, R is a commutative unital ring.

10pt **A1.** *Prove that $a \in \mathrm{Jac}(R)$ iff $1+ab$ is a unit for all $b \in R$.*

Solution. Let $a \in R$. For any ideal M, $a \in M$ implies that $1+ab \notin M$ for every $b \in R$; so, if a is contained in all maximal ideals of R, then $1+ab$ is not contained in any maximal ideal, so is not contained in any proper ideal of R, so is a unit. Conversely, if $a \notin M$ for some maximal ideal M, then, since R/M is a field, we have $ab = 1 \bmod M$ for some b, so $1-ab \in M$, so $1+a(-b)$ is not a unit.

7.4.37. *Let R be a commutative unital ring.*

5pt (a) *Prove that if R is local with maximal ideal M, then $M = \{a \in R : a \text{ is not a unit}\}$. Conversely, prove that if the nonunit elements of R form an ideal M, then R is local with maximal ideal M.*

Solution. Let R be local with maximal ideal M. If $a \in R \setminus M$ is not a unit, then (a) is a proper ideal of R, which by Zorn's lemma, is contained in a maximal ideal of R, that is, $a \in M$. Also, no unit u of R is contained in M, since $(u) = (1)$. Hence, M is the set of non-units of R.

Conversely, assume that the nonunit elements of R form an ideal M. If I is any proper ideal of R, then any $a \in I$ is not a unit, so $a \in M$, so $I \subseteq M$. Hence, M contains all other proper ideals of R, and is the only maximal ideal of R.

5pt (b) *Let P be a prime ideal in R, let $D = R \setminus P$; then D is a multiplicative set. If D contains no zero divisors, prove that $D^{-1}R$ is a local ring.*

Solution. D is multiplicative since $a, b \notin P$ implies that $ab \notin P$.

$D^{-1}P$ is a proper ideal since $\frac{a}{d} \neq 1$ for any $a \in P$, $d \in D$.

If $\frac{a}{d} \in D^{-1}R \setminus D^{-1}P$, then $a \notin P$, so $a \in D$, so $\frac{d}{a} \in D^{-1}R$, so $\frac{a}{d}$ is a unit.

On the other hand, if $\frac{a}{d} \in D^{-1}P$, then $\frac{a}{d}$ is not a unit: if $\frac{a}{d} \cdot \frac{b}{c} = 1$, then $ab = dc$ with $a \in P$ and $d, c \in D$, so $dc \in P$, which is impossible. Hence, the ideal $D^{-1}P$ is the set of all non-units of $D^{-1}R$, and so, is the single maximal ideal of this ring.

5pt **A2.** (a) *Prove that the union of two closed subsets of* $\mathrm{Spec}(R)$ *is closed.*

Solution. Let I and J be ideals in R, let $V_I = \{P \in \mathrm{Spec}(R) : I \subseteq P\}$ and $V_J = \{P \in \mathrm{Spec}(R) : J \subseteq P\}$. By the definition of prime ideals, for any prime ideal P we have $IJ \subseteq P$ iff $I \subseteq P$ or $J \subseteq P$. So, $V_I \cup V_J = V_{IJ}$ and is a closed subset of $\mathrm{Spec}(R)$.

5pt (b) *Prove that the intersection of any collection of closed subsets of* $\mathrm{Spec}(R)$ *is closed.*

Solution. Let I_α, $\alpha \in \Lambda$, be a family of ideals of R, and for every $\alpha \in \Lambda$ let $V_{I_\alpha} = \{P \in \mathrm{Spec}(R) : I_\alpha \subseteq P\}$. Let $I = \sum_{\alpha \in \Lambda} I_\alpha$; I claim that $\bigcap_{\alpha \in \Lambda} V_{I_\alpha} = V_I$ and so, is a closed subset of $\mathrm{Spec}(R)$. Indeed, a prime ideal P is contained in V_{I_α} for all α iff $P \supseteq I_\alpha$ for all α iff $P \supseteq I$.

10pt **A3.** *Let $\varphi : R \longrightarrow S$ be a homomorphism of commutative unital rings. Prove that the induced mapping $\eta : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ (defined by $\eta(P) = \varphi^{-1}(P)$) is continuous, that is, for any closed subset V of $\mathrm{Spec}(R)$, $\eta^{-1}(V)$ is a closed subset of* $\mathrm{Spec}(S)$.

Solution. Let $V = V_I$, where I is an ideal in R, that is, $V = \{\{Q \in \mathrm{Spec}(R) : I \subseteq Q\}$. For $P \in \mathrm{Spec}(S)$ we have $\eta(P) \in V$ iff $\varphi^{-1}(P) \supseteq I$ iff $P \supseteq \varphi(I)$. Let $J = (\varphi(I))$, the ideal in S generated by $\varphi(I)$; then $P \supseteq \varphi(I)$ iff $P \supseteq J$. This means that $\eta^{-1}(V) = \{P \in \mathrm{Spec}(S) : J \subseteq P\}$, a closed subset of $\mathrm{Spec}(S)$.

10pt **A4.** *If Q is an ideal of R such that the ideal $\mathrm{rad}(Q)$ is maximal, prove that Q is primary.*

Solution. Let $M = \mathrm{rad}(Q)$. After replacing R by R/Q we may assume that $Q = 0$ and $M = \mathrm{Nil}(R)$. Then M is the intersection of all the prime ideals of R; since M is maximal, M is the only prime ideal in R, R is a local ring, and all elements of $R \setminus M$ are units. So, R/Q has no zero divisors other than nilpotent elements, which means that Q is a primary ideal.

15pt **A5.** *If R is Noetherian, prove that any proper radical ideal in R is an intersection of finitely many prime ideals.*

Solution. Assume that the statement is not true. Then since R is Noetherian, the set of "proper radical ideals of R not representable as a finite intersection of prime ideals" has a maximal element I. Then I is not prime, so there are $a, b \in R$ such that $ab \in I$ whereas $a, b \notin I$. I claim that $\mathrm{rad}(I + (a)) \cap \mathrm{rad}(I + (b)) = I$. Indeed, clearly $I \subseteq \mathrm{rad}(I + (a))$ and $I \subseteq \mathrm{rad}(I + (b))$. If $c \in \mathrm{rad}(I + (a)) \cap \mathrm{rad}(I + (b))$, then $c^n \in I + (a)$ and $c^m \in I + (b)$ for some n, m, so $c^{n+m} \in I + I(a) + I(b) + (ab) \subseteq I$, so $c \in \mathrm{rad}(I) = I$. (It would be a little easier to factor R/I and assume that I is 0.)

Now, since $a, b \notin I$ and none of $I + (a)$, $I + (b)$ is a subideal of the other, $\mathrm{rad}(I + (a))$ and $\mathrm{rad}(I + (b))$ are proper radical ideals that are larger than I. So, $\mathrm{rad}(I + (a)) = \bigcap_{i=1}^{k} P_i$, $\mathrm{rad}(I + (b)) = \bigcap_{j=1}^{l} P'_j$ for some prime ideals P_i and P'_j, and $I = \left(\bigcap_{i=1}^{k} P_i\right) \cap \left(\bigcap_{j=1}^{l} P'_j\right)$.