Solutions to Homework 4 **Math 5590H**

5pt **3.1.25,27.** (a) *Prove that a subgroup N of G is normal iff $aNa^{-1} \subseteq N$ for all $a \in G$.*

Solution. If N is normal then $aNa^{-1} = N$ for all $a \in G$ by definition. Assume that $aNa^{-1} \subseteq N$ for all $a \in G$. Then for every $a \in G$, since also $a^{-1}N(a^{-1})^{-1} \subseteq N$, we have $N \subseteq aNa^{-1}$, and so, $aNa^{-1} = N$. Hence, $N \trianglelefteq G$.

5pt (b) *Let $G = \mathrm{GL}_2(\mathbb{Q})$, let N be the subgroup of upper triangular matrices with integer entries and 1-s on the main diagonal: $N = \left\{\begin{pmatrix}1&k\\0&1\end{pmatrix},\ k \in \mathbb{Z}\right\}$, and let $a = \begin{pmatrix}1&0\\0&1/2\end{pmatrix}$. Show that $aNa^{-1} \subseteq N$ but $aNa^{-1} \neq N$.*

Solution. For every $A = \begin{pmatrix}1&k\\0&1\end{pmatrix} \in N$ we have

$$aAa^{-1} = \begin{pmatrix}1&0\\0&1/2\end{pmatrix}\begin{pmatrix}1&k\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix} = \begin{pmatrix}1&2k\\0&1\end{pmatrix}.$$

So, aNa^{-1} is a proper subgroup of N, consisting of the matrices whose $(1,2)$-entry is even.

5pt (c) *Let N be a finite subgroup of a group G. Prove that if $aNa^{-1} \subseteq N$ for some $a \in G$, then $aNa^{-1} = N$.*

Solution. Conjugation by a is a bijection, so aNa^{-1} has the same cardinality as N. So, if N is finite and $aNa^{-1} \subseteq N$, then $aNa^{-1} = N$.

5pt **3.1.36.** *Prove that if $G/Z(G)$ is cyclic, then G is abelian.*

Solution. Let $Z = Z(G)$, and let $x \in G$ be such that $\overline{x} = xZ$ generates G/Z. Then all cosets of Z in G have form x^nZ, $n \in \mathbb{Z}$, so every $a \in G$ has form x^nz for some $n \in \mathbb{Z}$ and $z \in Z(G)$. Now, for any $a = x^nz$ and $b = x^mw$, where $z, w \in Z$, we have $ab = x^nzx^mw = x^{n+m}zw = x^{n+m}wz = x^mwx^nz = ba$, so G is commutative.

5pt **Cf. 3.1.39.** *If G is a non-abelian group, prove that the diagonal subgroup $D = \{(a,a) \mid a \in G\}$ of $G^2 = G \times G$ is not normal in G^2.*

Solution. Let $a, b \in G$ be non-commuting elements, so that $bab^{-1} \neq a$. Then $(1,b)(a,a)(1,b)^{-1} = (1a1^{-1}, bab^{-1}) = (a, bab^{-1}) \notin D$.

5pt **3.1.14(b).** *Show that each element of $\mathbb{Q}/\mathbb{Z}$ has a finite order but there are elements of arbitrarily large order.*

Solution. For any $x = m/n \in \mathbb{Q}$ we have $nx \in \mathbb{Z}$, so $n\bar{x} = 0$ in $\mathbb{Q}/\mathbb{Z}$ (where $\bar{x} = x + \mathbb{Z}$ is the coset of $\mathbb{Z}$ containing x), so $\bar{x}$ has a finite order. For $x = \frac{1}{d}$ with $d \in \mathbb{N}$, the minimal n for which $nx \in \mathbb{Z}$ and so, $n\bar{x} = 0$, is $n = d$, so there are elements of arbitrarily large orders in $\mathbb{Q}/\mathbb{Z}$.

3.1.34. *Let $D_{2n} = \langle r, s \,|\, r^n = s^2 = (rs)^2 = 1\rangle$ be the standard presentation of the dihedral group D_{2n} and let $k \,\big|\, n$.*

5pt (a) *Prove that $H = \langle r^k \rangle$ is a normal subgroup of D_{2n}.*

Solution. For any k, we have $rr^kr^{-1} = r^k$ and $sr^ks^{-1} = r^{-k}$, so $rHr^{-1} = H$ and $sHs^{-1} = H$. Since G is generated by r and s, this implies that $H \triangleleft G$.

5pt (b) *Prove that $D_{2n}/H \cong D_{2k}$.*

Solution. Informally, by factorizing D_{2n} by H we just add one more relation, $r^k = 1$, which implies and replaces $r^n = 1$ and makes $D_{2n}/H = \langle r, s \,|\, r^k = s^2 = (rs)^2 = 1\rangle \cong D_{2k}$. More formally, $D_{2n}/H = \{\overline{1}, \overline{r}, \overline{r}^2, \ldots, \overline{r}^{k-1}, \overline{s}, \overline{s}\cdot\overline{r}, \overline{s}\cdot\overline{r}^2, \ldots, \overline{s}\cdot\overline{r}^{k-1}\}$, and the multiplication table of this group is the same as in the group G_{2k}:

$$\overline{r}^i\overline{r}^j = \overline{r}^{i+j}, \quad (\overline{s}\cdot\overline{r}^i)\overline{r}^j = \overline{s}\cdot\overline{r}^{i+j}, \quad \overline{r}^i(\overline{s}\cdot\overline{r}^j) = \overline{s}\cdot\overline{r}^{j-i}, \quad \text{and } (\overline{s}\cdot\overline{r}^i)(\overline{s}\cdot\overline{r}^j) = \overline{r}^{j-i}$$

for all $i, j \in \{0, \ldots, k-1\}$, where $i + j$, $j - i$ are taken $\bmod\, k$.

5pt **3.1.42.** *Assume that H and K are normal subgroups of G with $H \cap K = 1$. Prove that H and K commute: $xy = yx$ for all $x \in H$ and $y \in K$.*

Solution. For any $x \in H$ and $y \in K$ we have $xyx^{-1} \in K$ so $[x, y] = xyx^{-1}y^{-1} \in K$, and also $yx^{-1}y^{-1} \in H$ so $[x, y] = xyx^{-1}y^{-1} \in H$. Thus $[x, y] \in H \cap K = 1$, so $[x, y] = 1$, so x and y commute.

5pt **3.2.5.** *Prove that if H is the only subgroup of order n in G, then $H \trianglelefteq G$.*

Solution. For any $a \in G$, since aHa^{-1} is a subgrop of order n in G, we have $aHa^{-1} = H$.