Solutions to Homework 5 **Math 5590H**

10pt **3.3.3.** *If H is a normal subgroup of a group G of prime index p, prove that for any $K \leq G$ either $K \leq H$, or $HK = G$ and $|K : (K \cap H)| = p$.*

Solution. Since $|G/H| = p$, we have $G/H \cong \mathbb{Z}_p$. Let $K \leq G$. Then HK is a subgroup of G and $HK/H \leq G/H$, so $|HK/H| \,\big|\, |G/H| = p$, so either $|HK/H| = 1$, in which case $HK = H$ and so $K \leq H$; or $|HK/H| = p$, in which case $HK = G$ and $|K : K \cap H| = |HK : H| = p$.

10pt **3.2.19.** *Prove that if N is a normal subgroup of a finite group G and* $\gcd(|N|, |G : N|) = 1$ *then N is the unique subgroup of G of order $|N|$.*

Solution. Let $K \subseteq G$ with $|K| = |N|$. Let $\pi\colon G \longrightarrow G/N$ be the projection homomorphism. Then $\pi(K) \leq G/N$ so $|\pi(K)| \,\big|\, |G/N| = |G : N|$. On the other hand, $\pi(K) \cong K/(K \cap N)$ is a factor group of K, so $|\pi(K)| \,\big|\, |K| = |N|$. So $|\pi(K)| = 1$, so $K \leq H$, so $K = H$.

10pt **3.3.7.** *Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.*

Solution. Consider the homomorphism $\varphi\colon G \longrightarrow (G/M) \times (G/N)$, $\varphi(a) = (a \bmod M, a \bmod N)$. (I should check that φ is a homomorphism, but this is straightforward.) To show that φ is surjective, let $B \in G/M$ and $C \in G/N$. Since $G = MN$, $B = bM$ for some $b \in N$, and $C = cN$ for some $c \in M$. Then for $a = bc$ we have $a \bmod M = b \bmod M$ and $a \bmod N = c \bmod N$, so $\varphi(a) = (B, C)$. Since $\operatorname{Ker} \varphi = M \cap N$, $M \cap N$ is normal and by the 1st isomorphism theorem, $G/(M \cap N) \cong (G/M) \times (G/N)$.

10pt **A1.** *Let F be a finite field of order q (that is, $F = \mathbb{F}_q$), let $n \in \mathbb{N}$, let N be the group $\{c \in F : c^n = 1\}$ of n-th roots of unity in F, let $|N| = d$. The special linear group $\mathrm{SL}_n(F)$ is the group of $n \times n$ matrices with determinant 1, the group $\mathrm{PSL}_n(F)$ is defined as $\mathrm{SL}_n(F)/Z\big(\mathrm{SL}_n(F)\big)$. Find the order of $\mathrm{SL}_n(F)$ and of $\mathrm{PSL}_n(F)$.*

Solution. $\mathrm{SL}_n(F)$ is the kernel of the homomorphism $\det\colon \mathrm{GL}_n(F) \longrightarrow F^*$. Since the homomorphism det is surjective, by the 1st isomorphism theorem, $\mathrm{GL}_n(F)/\mathrm{SL}_n(F) \cong F^*$, which has $q-1$ elements. It follows that $|\mathrm{SL}_n(F)| = |\mathrm{GL}_n(F)|/(q-1) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})/(q-1)$.

The center of $\mathrm{SL}_n(F)$ consists of scalar matrices cI with $c^n = 1$, that is, $c \in N$. Hence, $|\mathrm{PSL}_n(F)| = |\mathrm{SL}_n(F)|/d$.

A2. *Find a composition series for the groups*

5pt (a) Q_8.

Solution. $1 \trianglelefteq \langle -1 \rangle \trianglelefteq \langle i \rangle \trianglelefteq Q_8$, the factors are $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$.

5pt (b) D_8.

Solution. $1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$, the factors are $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$.

5pt (c) D_{12}.

Solution. $1 \trianglelefteq \langle r^3 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_{12}$, the factors are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$.

10pt **Cf. 3.4.8.** *Prove that a finite group is solvable iff it is polycyclic. Find (an infinite) solvable non-polycyclic group.*

Solution. Informally: every finite abelian group "is made" of cyclic groups, so if G is made of finite abelian groups then it is made of cyclic groups too. More formally: let G be solvable, let $1 = H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$ be a subnormal series with H_{i+1}/H_i being abelian for all i. Since for any i, H_{i+1}/H_i is a finite abelian group, in its composition series $1 = L_{i,0} \trianglelefteq L_{i,1} \trianglelefteq \cdots \trianglelefteq L_{i,m_i} = L_{i+1}/L_i$ the factors are simple abelian, so, cyclic. By the isomorphism theorems, we have a series $H_i = K_{i,0} \trianglelefteq K_{i,1} \trianglelefteq \cdots \trianglelefteq K_{i,m_i} = H_{i+1}$ with the same factors. Combining these series for all i, we get a subnormal series of G with cyclic factors.

$\mathbb{Q}$ is abelian, so solvable, but not polycyclic: indeed, any polycyclic group is finitely generated, but $\mathbb{Q}$ is not.

4.1.9. *Assume that a group G acts transitively on the finite set X and let H be a normal subgroup of G. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the distinct orbits of H in X.*

5pt (a) *Prove that G permutes the orbits $\mathcal{O}_i$: for every $a \in G$ and each i, $a\mathcal{O}_i = \mathcal{O}_j$ for some j. Prove that G acts transitively on the set $\{\mathcal{O}_1, \dots, \mathcal{O}_r\}$. Deduce that all the orbits $\mathcal{O}_i$ have the same cardinality.*

Solution. Let $\mathcal{O}$ and $\mathcal{O}'$ be two orbits under the action of H, $\mathcal{O} = Hx$ and $\mathcal{O}' = Hx'$ for some $x, x' \in X$. Find $a \in G$ such that $x' = ax$. (Such an a exists since G actis transitively on X.) Then, since H is normal, $\mathcal{O}' = Hax = aHx = a\mathcal{O}$. It follows that all orbits in X under the action of H have the same cardinality.

5pt (b) *Let $x \in X$ and $\mathcal{O} = Hx$. Prove that $|\mathcal{O}| = \big|H : (H \cap G_x)\big|$ (where G_x is the stabilizer of x in G) and that $r = |G : HG_x|$.*

Solution. Since $H_x = G_x \cap H$, we have $|\mathcal{O}| = |H : H_x| = \big|H : (H \cap G_x)\big|$. Next, by a counting principle, $\big|H : (H \cap G_x)\big| = |HG_x : G_x|$. Hence, by another counting principle, $r = |X|/|\mathcal{O}| = |G : G_x|/|HG_x : G_x| = |G : HG_x|$.

Or, alternatively: an element $a \in G$ fixes $\mathcal{O} = Hx$ iff $ax \in \mathcal{O}$, that is, $ax = hx$ for some $h \in H$. But this means that $a^{-1}h \in G_x$, that is, $a \in G_xH = HG_x$.