10pt **A1.** *Let H be a subgroup of a group G. Prove that $N = \bigcap_{a\in G} aHa^{-1}$ is normal in G and is the maximal subgroup of H that is normal in G. (That is, if $K \le H$ and $K \trianglelefteq G$, then $K \le N$.)*

Solution. N is normal in G since it is the kernel of the action of G on the set G/H of left cosets of H in G.

Or directly: for any $b \in G$,

$$bNb^{-1} = b\Big(\bigcap_{a\in G} aHa^{-1}\Big)b^{-1} = \bigcap_{a\in G} baHa^{-1}b^{-1} = \bigcap_{c\in G} cHc^{-1} = N$$

(since the set $\{ba,\ a \in G\} = G$).

Ok, using elements: Let $n \in N$. Let $b \in G$. For any $a \in G$ we have $n \in b^{-1}aHa^{-1}b$, so $n = b^{-1}aha^{-1}b$ for some $h \in H$, so $bnb^{-1} = aha^{-1} \in aHa^{-1}$. So, $bnb^{-1} \in \bigcap_{a\in G} aHa^{-1} = N$. So, $bNb^{-1} \le N$ for all $b \in G$.

On the other hand, let $K \le H$ be such that $K \trianglelefteq G$. Then for any $a \in G$, $aKa^{-1} = K$, so $K = \bigcap_{a\in G} aKa^{-1} \le \bigcap_{a\in G} aHa^{-1} = N$. (Or using elements: for any $c \in K$ and $a \in G$ we have $a^{-1}ca \in K$, so $c \in aKa^{-1} \subseteq aHa^{-1}$. Hence, $c \in \bigcap_{a\in G} aHa^{-1} = N$.)

5pt **A2.** (a) *If a group G act on a set X and N is the kernel of this action, show that the quotient group G/N also acts on X by $\overline{a}x = ax$, $x \in X$, $a \in G$.*

Solution. The action is (or induces) a homomorphism $G \longrightarrow S_X$ with kernel N, $a \mapsto \varphi_a$, so a homomorphism $G/N \longrightarrow S_X$ by $\overline{a} \mapsto \varphi_a$, $a \in G$.

5pt (b) *Let an action of a group G on a set X be transitive and such that for some $x \in X$ the stabilizer $N = G_x$ is a normal subgroup of G. Prove that N is the kernel of the action and that the induced action of G/N on X is regular.*

Solution. Since the action is transitive, its kernel is $\bigcap_{a\in G} aNa^{-1}$, which is equal to N since N is normal. Hence, the group G/N also acts on X, with the stabilizer $(G/N)_x$ being trivial; hence, this action is regular. (Every transitive action is isomorphic to the action by left multiplications on the set G/H where H is the stabilizer of a (any) point of X.)

5pt **4.2.4.** *Use the left regular representation of the group Q_8 to find two elements of S_8 that generate a group isomorphic to Q_8.*

Solution. i acts on G by left multiplication as the permutation $\sigma = (1, i, -1, -i)(j, k, -j, -k)$, and j by $\rho = (1, j, -1, -j)(i, -k, -i, k)$. Since i and j generate Q_8 and the left regular action induces an embedding $Q_8 \longrightarrow S_8$, σ and ρ generate a subgroup of S_8 isomorphic to Q_8.

5pt **4.2.5a.** *In the standard presentation for D_8, let $H = \langle s\rangle$. Enumerate the left cosets of H in G. Find the homomorphism $D_8 \longrightarrow S_4$ induced by the action of D_8 by left multiplications on the set D_8/H (of left cosets of H in D_8).*

Solution. Let's denote the left cosets of H in D_8 this way: $H = 1$, $rH = 2$, $r^2H = 3$, $r^3H = 4$. Then the action of r on the set of cosets is cyclic, $(1, 2, 3, 4)$, and of s is $(1)(2, 4)(3) = (2, 4)$ (since $sH = H$, $srH = r^3sH = r^3H$, etc.). That is, the action of r and s coincide with the standard action of these elements on the set $\{1, 2, 3, 4\}$ of the vertices of a square! Thus, the homomorphism $D_8 \longrightarrow S_4$ defined by this action is the standard embedding of D_8 into S_4.

5pt **A3.** *How many elements of S_6 commute with $\sigma = (1, 2)(3, 4, 5)$?*

Solution. σ has $\frac{6\cdot5}{2} \cdot \frac{4\cdot3\cdot2}{3} = 5!$ conjugates in S_6, so the number of elements in S_6 commuting with σ (that is, the cardinality of the centralizer of σ) is $6!/5! = 6$. (Hence, these only are $\tau^x\rho^y$, $x = 0, 1$, $y = 0, 1, 2$, where $\tau = (1, 2)$ and $\rho = (1, 2, 3)$.)

10pt **A4.** (a) *If finite groups A and B have coprime orders, prove that any subgroup of $A \times B$ has form $H \times K$ where $H \le A$ and $K \le B$.*

Solution. Let $G \le A \times B$. Let π_1 and π_2 be the projections from $A \times B$ onto A and B respectively, and let $H = \pi_1(G)$ and $K = \pi_2(G)$; Clearly, $G \le H \times K$. We also have $|H| \mid |G|$, $|K| \mid |G|$, so $\operatorname{lcm}(|H|, |K|) \mid |G|$; but $|H| \mid |A|$, $|K| \mid |B|$, and $|A|$, $|B|$ are coprime, so $|H|$ and $|K|$ are coprime, so $\operatorname{lcm}(|H|, |K|) = |H| \cdot |K|$; hence, $|G| = |H| \cdot |K| = |H \times K|$, and $G = H \times K$.

Another solution. Let $G \le A \times B$. Put $H = G \cap A$ and $K = G \cap B$; then $H \times K \le G$. Let $(a, b) \in G$. By the Chinese remainder theorem, there exists n such that $n = 1 \bmod |A|$ and $n = 0 \bmod |B|$; we then have $(a, b)^n = (a, 1) = a$, so $a \in H$. Similarly, $b = (1, b) \in K$. Hence, $G \le H \times K$.

5pt (b) *Give an example of a subgroup of $A \times B$, with $|A|$ and $|B|$ not coprime, NOT of the form $H \times K$ with $H \le A$ and $K \le B$.*

Solution. For $A = B$, the diagonal $\{(a, a),\ a \in A\}$ of $A \times B = A^2$ is a subgroup of A^2 not of the form $H \times K$.

10pt **A5.** *Prove that* $\mathrm{GL}_2(\mathbb{R}) \neq \mathrm{SL}_2(\mathbb{R}) \times C$*, whereas* $\mathrm{GL}_3(\mathbb{R}) = \mathrm{SL}_3(\mathbb{R}) \times C$*, where C is the subgroup of scalar matrices cI, $c \in \mathbb{R}^*$.*

Solution. In $\mathrm{GL}_2(\mathbb{R})$ the groups $\mathrm{SL}_2(\mathbb{R})$ and C have a nontrivial intersection: both contain the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

In $\mathrm{GL}_3(\mathbb{R})$, if $cI \in \mathrm{SL}_3(\mathbb{R})$ then $c^3 = 1$ so $c = 1$, hence $\mathrm{SL}_3(\mathbb{R}) \cap C = 1$. Also, for every $A \in \mathrm{GL}_3(\mathbb{R})$, $A = cIA'$ where $c = \sqrt[3]{\det A}$ and $A' = \frac{1}{c} A \in \mathrm{SL}_3(\mathbb{R})$, so $\mathrm{GL}_3(\mathbb{R}) = C\,\mathrm{SL}_3(\mathbb{R})$. And finally, $C = Z(\mathrm{GL}_3(\mathbb{R}))$, so its elements commute with the elements of $\mathrm{SL}_3(\mathbb{R})$.