5pt **5.4.13.** *Prove that for any $n \in \mathbb{N}$, $D_{8n}$ is not isomorphic to $D_{4n} \times \mathbb{Z}_2$.*

*Solution.* $D_{8n}$ contains an element of order $4n$ whereas $D_{4n} \times \mathbb{Z}_2$ does not. Indeed, the order of any element of $D_{4n}$ either is 2 or divides $2n$, so for any $(a,b) \in D_{4n} \times \mathbb{Z}_2$, if $|a|$ is even, then $|(a,b)| = |a| \leq 2n$, and if $|a|$ is odd, then $|a| \leq n$, so $|(a,b)| \leq 2n$.

5pt **A1.** *Prove that every element of the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}_n$ has finite order, and find an element of $\prod_{n=1}^{\infty} \mathbb{Z}_n$ of an infinite order.*

*Solution.* For any element $a = (a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots)$ of $\bigoplus_{n=1}^{\infty} \mathbb{Z}_n$ we have $k!a = 0$, so $a$ has finite order.

For the element $a = (1, 1, 1, \ldots)$, for any $k \in \mathbb{N}$, $ka = (k, k, k, \ldots) \neq 0$ since the $(k+1)$-st entry $k$ is nonzero in $\mathbb{Z}_{k+1}$.

10pt **A2.** *Let subgroups $H, K \leq G$ satisfy $HK = G$ and $hk = kh$ for all $h \in H$ and $k \in K$, and let $N = H \cap K$. Prove that $G \cong H *_N K$ under an isomorphism that "respects" $H$ and $K$: $h \leftrightarrow (h,1)$ and $k \leftrightarrow (1,k)$.*

*Solution.* Since $N \leq K$, $N$ centralizes $H$ (elements of $N$ commute with all elements of $H$). Since $N \leq H$, $N$ centrlizes $K$. Since $HK = G$, $N$ centralizes $G$, that is, $N \leq Z(G)$, so $N \leq Z(H)$ and $N \leq Z(K)$.

The embedding homomorphisms from $N$ to $H$ and $K$ define the central product $H *_N K = (H \times K)/D$ where $D = \{(a, a^{-1}) : a \in N\}$. Define a homomorphism $\varphi\colon H \times K \longrightarrow G$ by $\varphi(h,k) = hk$, $h \in H$, $k \in K$. Since $G = HK$, $\varphi$ is surjective. We have $\varphi(h,k) = 1$ iff $hk = 1$ iff $h = k^{-1} \in H \cap K = N$, that is, $\ker(\varphi) = D$. By the 1-st isomorphism theorem, $\varphi$ induces an isomorphism $G \cong (H \times K)/D$. And, under $\varphi$, $(h,1) \leftrightarrow h$ for any $h \in H$ and $(1,k) \leftrightarrow k$ for any $k \in K$.

10pt **A3.** *Let $n, m \in \mathbb{N}$, and let $d = \gcd(n,m)$ and $l = \operatorname{lcm}(n,m)$. Prove that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_l \times \mathbb{Z}_d$.*

*Solution.* Let $n = p_1^{r_1} \cdots p_k^{r_k}$ and $m = p_1^{s_1} \cdots p_k^{s_k}$ where $p_i$ are distinct primes and some of $r_i$, $s_i$ may be equal to 0. Then $d = p_1^{\min\{r_1,s_1\}} \cdots p_k^{\min\{r_k,s_k\}}$ and $l = p_1^{\max\{r_1,s_1\}} \cdots p_k^{\max\{r_k,s_k\}}$. By the Chinese remainder theorem,

$$\begin{aligned}
\mathbb{Z}_n \times \mathbb{Z}_m &\cong \left(\mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}\right) \times \left(\mathbb{Z}_{p_1^{s_1}} \times \cdots \times \mathbb{Z}_{p_k^{s_k}}\right) \cong \left(\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_1^{s_1}}\right) \times \cdots \times \left(\mathbb{Z}_{p_k^{r_k}} \times \mathbb{Z}_{p_k^{s_k}}\right) \\
&\cong \left(\mathbb{Z}_{p_1^{\max\{r_1,s_1\}}} \times \mathbb{Z}_{p_1^{\min\{r_1,s_1\}}}\right) \times \cdots \times \left(\mathbb{Z}_{p_k^{\max\{r_k,s_k\}}} \times \mathbb{Z}_{p_k^{\min\{r_k,s_k\}}}\right) \\
&\cong \left(\mathbb{Z}_{p_1^{\max\{r_1,s_1\}}} \times \cdots \times \mathbb{Z}_{p_k^{\max\{r_k,s_k\}}}\right) \times \left(\mathbb{Z}_{p_1^{\min\{r_1,s_1\}}} \times \cdots \times \mathbb{Z}_{p_k^{\min\{r_k,s_k\}}}\right) \cong \mathbb{Z}_l \times \mathbb{Z}_d.
\end{aligned}$$

(Which is, by the way, the "invariant factors" decomposition of this group.)

10pt **A4.** *Let $n, m \in \mathbb{N}$, $d = \gcd(n,m)$, $l = \operatorname{lcm}(n,m)$. Then $\mathbb{Z}_d$ is a common factor of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Prove that $\mathbb{Z}_n \times_{\mathbb{Z}_d} \mathbb{Z}_m \cong \mathbb{Z}_l$.*

*Solution.* The subgroup $H = \mathbb{Z}_n \times_{\mathbb{Z}_d} \mathbb{Z}_m$ of $\mathbb{Z}_n \times \mathbb{Z}_m$ has order $nm/d = l$. (Indeed, given any $a \in \mathbb{Z}_n$, there are $m/d$ elements $b \in \mathbb{Z}_m$ such that $b \bmod d = a \bmod d$.) On the other hand, the element $(1,1)$ of $H$ has order $l$. Hence, $(1,1)$ generates $H$, and $H \cong \mathbb{Z}_l$.

**5.2.2,3(a,b,c).** *Give the list of elementary divisors and the invariant factors of all abelian groups of the order:*

5pt (a) $270 = 2 \cdot 3^3 \cdot 5$

*Solution.* The possible collections of elementary divisors of groups of this order are $(2, 3^3, 5)$, $(2, 3^2, 3, 5)$, and $(2, 3, 3, 3, 5)$; the corresponding collections of invariant factors are $(270)$, $(90, 3)$, and $(30, 3, 3)$.

5pt (b) $9801 = 3^4 \cdot 11^2$

*Solution.* The possible collections of elementary divisors are

$(3^4, 11^2),\ (3^3, 3, 11^2),\ (3^2, 3^2, 11^2),\ (3^2, 3, 3, 11^2),\ (3, 3, 3, 3, 11^2),$
$(3^4, 11, 11),\ (3^3, 3, 11, 11),\ (3^2, 3^2, 11, 11),\ (3^2, 3, 3, 11, 11),\ (3, 3, 3, 3, 11, 11);$

the corresponding collections of invariant factors are

$(3^4 \cdot 11^2),\ (3^3 \cdot 11^2, 3),\ (3^2 \cdot 11^2, 3^2),\ (3^2 \cdot 11^2, 3, 3),\ (3 \cdot 11^2, 3, 3, 3),$
$(3^4 \cdot 11, 11),\ (3^3 \cdot 11, 3 \cdot 11),\ (3^2 \cdot 11, 3^2 \cdot 11),\ (3^2 \cdot 11, 3 \cdot 11, 3),\ (3 \cdot 11, 3 \cdot 11, 3, 3).$

5pt (c) $320 = 2^6 \cdot 5$

*Solution.* The possible collections of elementary divisors are

$(2^6, 5),\ (2^5, 2, 5),\ (2^4, 2^2, 5),\ (2^4, 2, 2, 5),\ (2^3, 2^3, 5),\ (2^3, 2^2, 2, 5),\ (2^3, 2, 2, 2, 5),$
$(2^2, 2^2, 2^2, 5),\ (2^2, 2^2, 2, 2, 5),\ (2^2, 2, 2, 2, 2, 5),\ (2, 2, 2, 2, 2, 2, 5);$

the corresponding collections of invariant factors are

$(2^6 \cdot 5),\ (2^5 \cdot 5, 2),\ (2^4 \cdot 5, 2^2),\ (2^4 \cdot 5, 2, 2),\ (2^3 \cdot 5, 2^3),\ (2^3 \cdot 5, 2^2, 2),\ (2^3 \cdot 5, 2, 2, 2),$
$(2^2 \cdot 5, 2^2, 2^2),\ (2^2 \cdot 5, 2^2, 2, 2),\ (2^2 \cdot 5, 2, 2, 2, 2),\ (2 \cdot 5, 2, 2, 2, 2, 2).$

5pt **5.2.4(b).** *Determine which pairs of abelian groups listed are isomorphic (where the expression* $[n_1, \ldots, n_k]$ *denotes the group* $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$*):*
$[2^2, 2 \cdot 3^2],\quad [2^2 \cdot 3, 2 \cdot 3],\quad [2^3 \cdot 3^2],\quad [2^2 \cdot 3^2, 2].$

*Solution.* In the "elementary divisors" form these groups are, respectively, $[2^2, 2, 3^2]$, $[2^2, 3, 2, 3]$, $[2^3, 3^2]$, and $[2^2, 3^2, 2]$. Thus, only the 1st and the 4th of these groups are isomorphic.