Solutions to Homework 8 **Math 5590H**

10pt **A1.** *We have* $\mathrm{Aut}(D_8) = \{\varphi_{k,l},\ k \in \mathbb{Z}_4^*,\ l \in \mathbb{Z}_4\}$, *where* $\varphi_{k,l} : \left| \begin{smallmatrix} r \mapsto r^k \\ s \mapsto sr^l \end{smallmatrix} \right.$. *Find the group* $\mathrm{Inn}(D_8)$ *of inner autormorphisms of* D_8 *and find an outer automorphism of* D_8.

Solution. Since $Z(D_8) = \{1, r^2\}$, the group $\mathrm{Inn}(D_8) \cong D_8/Z(D_8) \cong V_4$, and cosists of $1 = \varphi_{1,0}$, the conjugation by r: $\begin{vmatrix} r \mapsto r \\ s \mapsto rsr^{-1} = sr^2 \end{vmatrix} = \varphi_{1,2}$, the conjugation by s: $\begin{vmatrix} r \mapsto srs^{-1} = r^3 \\ s \mapsto s \end{vmatrix} = \varphi_{3,0}$, and their product, the conjugation by sr: $\begin{vmatrix} r \mapsto r^3 \\ s \mapsto sr^2 \end{vmatrix} = \varphi_{3,2}$. Any other automorphism of Q_8, say, $\varphi_{1,1} : \begin{vmatrix} r \mapsto r \\ s \mapsto sr \end{vmatrix}$, is outer.

10pt **A2.** *Prove that* $\mathrm{Aut}(Q_8) \cong S_4$.

Solution. Let R be the group of rotations of the cube; we have $|R| = 24$. Let's label the faces of the cube with the symbols $i, -i, j, -j, k, -k$ so that each letter and its negative are on the opposite faces; then R is identified with a subgroup of the group $S_X \cong S_6$ of permutations of the set $X = \{i, -i, j, -j, k, -k\}$. The group $\mathrm{Aut}(Q_8)$ also acts faithfully as a group of permutations of X; we need to show that $H = R$. Every pair of elements $a, b \in X$ with $a \neq \pm b$ generates Q_8 and satisfies the same relations as i, j; so, there is $\varphi_{a,b} \in H$ with $\varphi_{a,b}(i) = a$ and $\varphi_{a,b}(j) = b$. Hence, $|H| = 6 \cdot 4 = 24$. Also, every automorphism $\varphi_{a,b}$ is in R: for any $a \in X$ there is a rotation of the cube that maps $i \mapsto a$; j is mapped to one of the neighboring faces, let's call it c; then there is a rotation, with respect to the axis $(a, -a)$, that preserves a and maps c to any face $b \neq \pm a$. Hence, $H \leq R$; since $|H| = |R|$, we have $H = R \cong S_4$.

4.4.8. *Let* $H \leq K \leq G$.

10pt (a) *Prove that if* $H \operatorname{char} K$ *and* $K \operatorname{char} G$, *then* $H \operatorname{char} G$. *Use this result to prove that the subgroup* $V = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ *is characteristic in* S_4.

Solution. Since $K \operatorname{char} G$, for any $\varphi \in \mathrm{Aut}(G)$ we have $\varphi(K) = K$, so $\varphi|_K$ is an automorphism of K. Since $H \operatorname{char} K$, we then have $\varphi|_K(H) = H$, that is, $\varphi(H) = H$. Hence, $H \operatorname{char} G$.

V is the set of all elemens of A_4 of order 1 and 2, thus $V \operatorname{char} A_4$. A_4 is the only subgroup of S_4 of index 2, so $A_4 \operatorname{char} S_4$. (Indeed, if $K \leq S_4$ with $|S_4 : K| = 2$, then for any $\sigma \in S_4$ we have $\sigma^2 \in K$, so $A_4 \leq K$, so $K = A_4$.) Hence, $V \operatorname{char} S_4$.

5pt (b) *Prove that if* $H \operatorname{char} K$ *and* $K \trianglelefteq G$ *then* $H \trianglelefteq G$.

Solution. For any $a \in G$, the inner automorphism (the conjugation) $\varphi_a(x) = axa^{-1}$ preserves K, $\varphi_a(K) = K$. Next, φ_a acts on K as an automorphism of K, so it preserves H, $\varphi_a(H) = H$. Hence, H is normal in G.

5pt (c) *Give an example to show that if* $H \trianglelefteq K$ *and* $K \operatorname{char} G$ *then* H *need not be normal in* G.

Solution. Define $G = \langle a, b, c \,|\, a^3 = b^3 = c^2 = 1,\ ab = ba,\ ca = bc,\ cb = ac \rangle$ of order 18. ($G = K \rtimes C$ where $K = \langle a, b \rangle \cong \mathbb{Z}_3^2$ and $C = \langle c \rangle \cong \mathbb{Z}_2$; c acts on K by switching a and b.) K is a characteristic subgroup of G, since it consists of all elements of order 3 in G (and the identity). The subgroup $H = \langle a \rangle$ is normal in K, since K is abelian. However H is not normal in G, since $cHc^{-1} = \langle b \rangle$.

Another solution. The subgroup $H = \{1, (1,2)(3,4)\}$ is normal in the group $K = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ (since K is abelian). By (a), K is a characteristic subgroup of S_4. However H is not normal in S_4 (since doesn't contain conjugates of its generator).

10pt **4.4.13.** *Let* G *be a group of order* $203 = 7 \cdot 29$ *and* H *be a normal subgroup of* G *of order* 7. *Prove that* $H \leq Z(G)$. *Deduce that* G *is abelian.*

Solution. If $H \trianglelefteq G$, then G acts on H by conjugations, which defines a homomorphism $\varphi : G \longrightarrow \mathrm{Aut}(H)$. $H \cong \mathbb{Z}_7$, so $\mathrm{Aut}(H) \cong \mathbb{Z}_7^*$ and has order 6. Since G and $\mathrm{Aut}(H)$ have coprime orders, φ is trivial, so all elements of G commute with the elements of H, so $H \leq Z(G)$.

Now, $|G| = 203 = 7 \cdot 29$, so either $Z(G) = G$ (in which case G is abelian), or $Z(G) = H$ and $G/Z(G) = 29$. But 29 is prime, so $G/Z(G)$ is cyclic, and by exercise 3.1.36, G is abelian in this case too.

10pt **A3.** *Construct a nonabelian group of order* 39; *give a presentation of this group in terms of generators and relations.*

Solution. We can construct the group G as a semidirect product $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$. To this end, we need a nontrivial homomorphism $\mathbb{Z}_3 \longrightarrow \mathrm{Aut}(\mathbb{Z}_{13}) \cong \mathbb{Z}_{13}^*$, that is, an element of order 3 in the group $\mathbb{Z}_{13}^*$. Taking powers of 2 we get $2, 4, 8, 3, 6, 12 = -1$, so 2 has order 12; hence, $2^4 = 3$ has order 3. (And indeed, $3^3 = 27 = 1$ in $\mathbb{Z}_{13}$.) Hence, we can take $H = \langle a \mid a^{13} = 1 \rangle \cong \mathbb{Z}_{13}$, $K = \langle b \mid b^3 = 1 \rangle \cong \mathbb{Z}_3$, $\varphi\colon K \longrightarrow \mathrm{Aut}(H)$ with $\varphi(b)(a) = a^3$, and get

$$G = H \rtimes_\varphi K = \langle a, b \mid a^{13} = b^3 = 1,\ bab^{-1} = a^3 \rangle.$$

10pt **5.5.9.** *The matrix* $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 4 \end{smallmatrix}\right)$ *has order* 5 *in* $\mathrm{GL}_2(\mathbb{F}_{19})$. *Use it to construct a nonabelian group of order* $1805 = 19^2 \cdot 5$.

Solution. We can construct the group G as a semidirect product of $H = \langle a, b \mid a^{19} = b^{19} = 1,\ ab = ba \rangle \cong \mathbb{Z}_{19}^2 = \mathbb{F}_{19}^2$, and $K = \langle c \mid c^5 = 1 \rangle \cong \mathbb{Z}_5$, with the action of c on H by conjugations defined by the matrix A:

$$G = \langle a, b, c \mid a^{19} = b^{19} = c^5 = 1,\ ba = ab,\ cac^{-1} = b,\ cbc^{-1} = a^{-1}b^4 \rangle.$$